Explaining Credit Spreads and Volatility Smirk: A Unified Framework*

Du Du
Hong Kong University of Science and Technology
Redouane Elkamhi
University of Toronto
September 2, 2011

Abstract

We demonstrate that incorporating severe economic conditions in a consumption-based equilibrium model can jointly explain empirical facts in both credit and option markets. Model-implied credit spreads and smirk premia respond monotonically to the severity of the Peso problem\(^1\). However, while the credit spread is countercyclical, the smirk is a poor proxy for economic states. Our model captures observed levels and volatility of credit spreads at different maturities, and generates desirable shapes of the spread term structure with a good time series fit. Simultaneously, it matches the volatility smirk for index options and the first two moments of government bonds and aggregate stock.

JEL code: C60, G12, G13

Key words: credit spread, default risk premium, smirk premium, habit formation, peso problem

\(^{1}\)We acknowledge the financial support from the Hong Kong RGC Research Grant (HKUST 641208).
I Introduction

By now, there is abundant empirical evidence that there is a link between options, equity and credit markets\(^2\). Recent literature has advanced our knowledge on how equilibrium structural models explain puzzles in each of these markets separately. Chen, Collin-Dufresne and Goldstein (2009, CCDG), Chen (2010), and Bhamra, Kuhn and Strebulaev (2010, BKS) are rare examples of models capable of resolving some facets of the credit-spread puzzles. Similarly, Benzoni, Collin-Dufresne, Goldstein (2010, BCDG), and Du (2011) are few exceptions of models that generate observed excess volatility and smirk patterns in the data. Since all these studies differ in their setups, an important question remains: What is the economic foundation sufficient to price and relate all these markets within one unified framework?

This paper aims to fill this gap in the literature. We recognize that in the absence of arbitrage, there exists a pricing kernel that determines prices of all assets (Harisson and Kreps (1979)). We use a representative agent model to jointly price defaultable bonds, aggregate equity, and equity index options. We assume a non-time-separable preference induced by habit formation, and the lognormal consumption process subject to a small-probability jump that models economic disaster (e.g., Barro (2006)). The implied pricing kernel thus commands two types of risks: the diffusive risks and the jump risks. We relate default rates for investment grade (IG) corporate bonds and loss rates to bad economic states proxied by consumption jumps. Our model can jointly capture the observed level and volatility of credit spreads demanded by investors to hold IG bonds for different maturities. It generates desirable shapes for the credit term structure and provides a good fit for the time-series variations. Simultaneously, it matches the aggregate stock market behavior and the volatility smirk for index options, and uncovers interesting links between the volatility smirk and credit spreads.

---

\(^2\)Cao, Yu and Zhong (2010) shows that implied volatility has an explanatory power for credit spread beyond historical volatility. Zhang, Zhou, and Zhu (2009) shows that in addition to historical volatility, jump risk have large explanatory power of default risk. Collin-Dufresne, Goldstein and Martin (2009; CGM) use a smirk premium as a proxy for unobservable jumps in asset values and show that it has a strong explanatory power for spreads. Further, Wang, Zhou and Zhou (2010) show that in addition to volatility being important for the price of default protection, the variance risk premium is a key determinant of firm-level credit spreads. Recently, Cremers, Driessen, and Maenhout (2008, CDM) finds that incorporating option-implied jump risk premia brings the implied credit spread levels much closer to observed levels. Carr and Wu (2009) develops and tests no-arbitrage link between a CDS and put options for individual firms.
The mechanism of the model is as follows. First, in the presence of habit formation, the agent’s risk aversion reacts negatively to changes in aggregate consumption creating an extra channel by which consumption innovations induce extra diffusions and jumps in agent’s marginal utility. This effect gives rise to high prices in both diffusive and jump risks, hence the high equity premium. Since deep out-of-the-money (OTM) puts are more sensitive to jumps than are at-the-money (ATMs), jump premium translates into the high smirk premium implicit in option data. Second, default induces dramatic downward jumps in bond prices implying the high quantity of default risks. Given the countercyclical default losses, this risk exposure peaks during bad times when risk prices are high. On the other hand, the price of jump risks translates into the price of default risks when defaults are associated with consumption jumps – the channel of relating default to the pricing kernel in our framework. Taken together, the high default loss and the high price of risks during disaster time depress valuations of IG bonds even though their default probabilities are low, hence the high credit spreads.

It is well known that countercyclical interest rates, as in habit formation models (e.g., Wachter, (2006)), lead to upward-sloping yield curves, while procyclical interest rates, as in long-run risk models (e.g., Bansal and Yaron, (2004)), lead to downward-sloping yield curves. Partly for this reason, habit formation models tend to underestimate while long-run risk models tend to over-estimate credit spreads for long-term defaultable bonds. The opposite pattern holds for short-term bonds. For example, Chen (2010) reports a 10-year spread of 45bp for Aaa firms, while the average spread from a 5-year credit default swap (CDS) written on such firms is around 10bp. Alternatively, using only habit formation, CCDG report a 4-year Aaa spread of 1bp. In our setup, the peso problem component strengthens precautionary saving. While

---

3To understand the different implications regarding credit spreads from habit and long-run risk, let’s equate the yield from the defaultable bonds in the two models. Since cash flows from defaultable bonds realize earlier (due to potential defaults before maturity) than those from default-free bonds, the respective yield term structure of the two models implies higher (lower) default-free yields in habit (long-run risk). Consequently, the habit model generates a lower credit spread than that of the long-run risk model.

4We collect data on CDS premium from Markit database. We apply the common filter in the literature (Dollar denominated, Senior, North American and Modified Restructuring event). The average 5-years CDS spread for the set of firms that maintained Aaa rating from January 2001 to December 2007 is 7bp. During the 2007 to 2008 large illiquidity shock, the spread increases dramatically. From December 2008 to March 2011, the average spread drops again to 13bp. In what follows we use 10bp as an average spread for Aaa bonds.
this makes interest rate less countercyclical, the interest rate term structure generally remains upward-sloping. Therefore, our model has the potential to capture the entire term structure of credit spreads.

We use Du (2011)'s estimation for consumption and habit formation processes based on index option data that is "out-of-sample" for credit spreads. For stochastic default loss, we impose that it is countercyclical, which is consistent with various studies (e.g., Shleifer and Vishney (1992) and Acharya et al (2009)). At the five-year contract horizon, the calibrated model generates around 181bp for Baa-Treasury spread and 10.3bp for Aaa-Treasury spread. These levels are very close to the observed CDS spreads for Baa- and Aaa- rated firms (153bp and 10bp), and they are lower than respective observed corporate bonds spreads. The reason for the latter discrepancy is that our model only accounts for credit risk, while it is widely acknowledged that other factors such as liquidity play important roles in corporate bond pricing. We thus interpret our matches for CDS averages and the underestimation for investment-grade bonds to be the strengths of our model.

Focusing on the Baa-Aaa to mitigate other non-default components, we find that the level and the standard deviation of the four-year Baa-Aaa spreads are 113bp and 46bp, respectively. These numbers are very close to the observed levels of 109bp and 41bp, and they improve on matches implied from both traditional structural models (e.g., Huang and Huang (2003)) and the more recent general equilibrium models (e.g., Chen (2010) and BKS (2010)). Simultaneously, our model generates the observed 6%-8% equity premium, 16%-18% volatility implied from ATM options, and 10% smirk premium between deep OTM and ATM options.

To provide implications on spread term structure, we calibrate our model to the term structure of objective default probabilities from Moody's. The short term spread is strictly positive with a level close to the data. Accounting for small probability consumption jumps coupled with a time-varying risk aversion induces a higher and stochastic risk-adjusted Q-intensity of default, which in turn induces a sizable short term spread. In addition, our model generates both upward- and hump-shaped spread term structures, depending on which objective default probabilities the model has been calibrated to. This is an empirical fact also documented in Jarrow et al (1997), which shows that the spread term structures are upward sloping for credits rated A or better. The spreads become slightly humped for Baa-rated credits. Finally, our model-implied term structure tends to flatten in the long run. This is also a desirable
feature observed in the data, which traditional structural models have difficulties generating due to the stationarity of default boundaries.

Next, we investigate model’s implications about time-series variations of 4-year Baa-Aaa spreads. We obtain innovations of the historical log consumption growth time series, which we use to construct the time series of the surplus consumption ratio and loss rate. We then compute the model-implied spreads and compare them with historical data. For the 92-year period between 1919–2010, we find that the two time series exhibit similar dynamics. Quantitatively, the model values mimic well the historical mean (130bp vs 126bp), the standard deviation (83.7bp vs 80.0bp), the minimum (46.5bp vs 34bp), and the maximum (460bp vs 510bp). The model-implied spreads correlate with the historical spreads by 62%, and the correlations of the changes of the two spreads is 23.3% for the full sample.

Besides matches in separate markets, our model also uncovers an intriguing link between the smirk premium and credit spreads in terms of their sensitivities to the peso problem. We know from Du (2011) that habit formation in the absence of a peso component can still produce a positive smirk premium, but the level is off by 50%. In addition to confirming this finding, we show that in the absence of the peso component, the implied credit spread is less than one-fourth of the observed level. Both the credit spread and smirk rise in lockstep as we increase the consumption jump severity until they reach their respective empirical level at our base case consumption jump calibration. Thus, we argue that a small disaster component is crucial to provide a link between the credit and the option markets. The intuition is straightforward: index options constitute the prime liquid market for insurance against systematic jumps, precisely the type of jumps that holders of defaultable bonds are exposed to. This finding improves on that of, for example, CDM, among others, who find that the information from option market is helpful to price corporate bonds.

To further understand the relationship between the smirk premium and credit spread, we investigate their comovement with respect to different states of the economy as proxied by different levels of surplus ratio $S$. Since defaults in our setup are related to severe macro economic conditions, our model naturally produces a countercyclical credit spread. The smirk premium, however, is a different story: it is

---

5Surplus ratio is defined as the ratio between current consumption minus a slow-moving weighted average of past consumption, and current consumption. It is the fundamental state variable in habit formation models.

6Credit spreads contain both an expected loss and risk premium components. The expected loss
decreasing at very low $S$, and becomes increasing for higher $S$. This is because an increase of the surplus ratio has two offsetting effects on the implied smirk premium. First, it decreases the severity of consumption jumps under the pricing ($Q$) measure, which gives the agent less incentives to hedge, hence the lower option premium for the given diffusive stock volatility $\sigma_{Ps}$. Second, it decreases $\sigma_{Ps}$, which increases the relative importance of jumps in the total stock price variations. As a result, the agent has stronger incentives to buy insurance from OTM puts, which drives up the smirk premium. The second effect dominates only when $S$ is relatively large. This insight suggests that it is not accurate to use the smirk premium to proxy for macroeconomic states. It also raises caution about running regressions of credit spreads on option premium, since i) they are both endogenously determined through the exposure of their respective market to economic disasters and ii) their relationship depends upon the economic states that the sample covers.

As explained above, our model’s resolution of credit spread puzzle hinges on a peso component, which drives jumps in the pricing kernel, together with relating defaults of IG bonds to these severe consumption shocks. The magnitude of consumption jumps used in this paper, hence the implied magnitude of pricing kernel jumps, is estimated by Du (2011) using the S&P 500 index option data in a nearly 20-year period. In our setup, the pricing kernel jump magnitude also controls the price of default risks. Thus, it generates the wedge in default intensity between the risk-neutral measure and the actual measure. We find that the default intensity ratio under the two measures is about 3.1, which is in line with several empirical studies (e.g., Berndt, Duffie, Douglas, Ferguson and Schranz (2008) and Elkamhi and Ericsson (2009)). This finding provide further evidence about links between the option and credit markets.

Finally, this paper contributes by bringing habit formation into the class of credit risk models with closed-form valuations for any types of bonds. Most existing models either study strategic default in which equity-holders make bankruptcy decisions or treat default as the first time when the firm value hits some exogenous boundary. component is monotonically related to loss rate, which rises as the states of the economy worsens. However, as explained in footnote 14, the quantity of systematic risk decreases as $S$ goes to zero leading to a reduction in risk premium. The fact that the first channel dominates for low level of $S$ ensures the intuitive and desirable result of countercyclical spread.

\footnote{While we focus our analysis on zero coupon bond, the closed-form valuation technique, which is based on the general framework of "linear factor processes augmented with structural changes" (LS) proposed by Du and Elkamhi (2011), can be easily extended to price any other type of bonds including the consol bond that makes continuous coupon payment for infinite time.}
Introducing market-wide variations adds to the state variable, making the problem intractable. In this paper, we assume that the investor has the same information set as the market – incomplete knowledge of the firm’s fundamental as in, among others, Jarrow, Lando, Turnbull (1997), Duffie and Singleton (1999), Duffie and Lando (2001), Giesecke (2004). Given the scope of our study, this approach offers the following advantages. First, it can be tractably embedded into the habit formation framework, which is known to price well equity and index options. Second, bond cash flows at the event of default are based only on observable, which are easy to calibrate. Third, the tractability of the model for bonds of any type greatly facilitates the study of default risks in relation to premiums implicit in other assets. In comparison, among the few tractable credit risk models (e.g., BKS (2010) and Chen (2010)), the researchers focus on consol bonds, and it is not apparent how their approaches can be applied to price other types of bonds with tractability.

The remainder of the paper is organized as follows. We introduce the model in Section II. Section III derives equilibrium asset prices. In Section IV, we describe the model mechanism. In Section V, we present our quantitative results. Section VI contains robustness exercises, and Section VII concludes.

II The model

A. Preference and pricing kernel

Time is continuous and infinite, and the uncertainty is represented by a complete probability space \((\Omega, \mathcal{F}, P)\) and an information filtration \((\mathcal{F}_t)_{t \geq 0}\), where \(\mathcal{F}_t\) denotes the information set observed up to period \(t\). A representative agent in the economy maximizes

\[
E \left[ \int_0^\infty e^{-\rho t} u(C_t, H_t) dt \right] = E \left[ \int_0^\infty e^{-\rho t} \frac{(C_t - H_t)^{1-\gamma}}{1-\gamma} dt \right]
\]

where \(C_t\) denotes the aggregate consumption; \(H_t\) denotes the habit level and \(\rho\) denotes the subjective time-discount rate. As in both Campbell and Cochrane (1996; CC) and Menzly, Santos and Veronesi (2004; MSV), we adopt the external habit formation specification in that the agent’s habit level is determined by the aggregate consumption rather than by his own consumption in the past. Since habit is external, the local curvature of the utility function, \(\gamma_t\), which measures the agent’s instantaneous
risk aversion degree, is given by

$$\gamma_t \equiv -\frac{C_t u_{cc}(C_t, H_t)}{u_c(C_t, H_t)} = \frac{C_t}{C_t - H_t} = \frac{1}{S_t},$$

where \( S_t \equiv \frac{C_t - H_t}{C_t} \) denotes the surplus consumption ratio. \( S_t \) is procyclical in that a low \( S \) implies a bad economic state, under which \( C_t \) is close to \( H_t \). As a result, \( \gamma_t \) is countercyclical implying that the agent is more risk averse during bad times. In particular, \( \gamma_t \) tends toward infinity as \( S_t \) goes to zero.

Following MSV, we assume that \( \gamma_t \) follows a mean reverting process, perfectly negatively correlated with innovations in log consumption, i.e.,

$$d\gamma_t = k(\bar{\gamma} - \gamma_t)dt - \alpha(\gamma_t - \beta)(dc_t - E_t[dc_t]),$$

where \( \bar{\gamma} \) is the long run average of the agent’s risk aversion; \( k \) controls the speed of mean reversion; \( c_t \) denotes the log consumption; \( \alpha > 0 \) captures the sensitivity of \( \gamma_t \) to consumption innovations; \( \beta \geq 1 \) sets the lower bound for \( \gamma_t \), and hence the upper bound for \( S_t \). Due to their monotone relations, either \( S_t \) or \( \gamma_t \) serves as the equivalent state variables in this model. Their variations naturally translate into variations in the prices and returns of the financial assets.

Following Wachter (2009) and Du (2011), we assume that the growth rate of the log consumption evolves as a random walk subject to a small-probability negative jump, i.e.,

$$dc_t \equiv d\log(C_t) = \mu_t dt + \sigma dB_t + b dN_t,$$

where \( B_t \) is a standard Brownian motion; \( N_t \) is a Poisson process capturing the random arrival of a small-probability economic disaster with the constant intensity \( \lambda \); \( b < 0 \) denotes the jump size of \( c_t \): upon the occurrence of the \( i \)-th disaster at \( \tau_i \), log consumption jumps from \( c(\tau_i) \) to \( c(\tau_i) + b \). For simplicity, we treat \( b \) as a constant, and model implications are largely unchanged at the random consumption jump sizes. By Ito’s lemma with jumps (e.g., Appendix F of Duffie, 2001),

$$\frac{dC_t}{C_t} = \mu dt + \sigma dB_t + J_C dN_t,$$

where \( J_C \equiv e^b - 1 \) denotes the consumption jump size; \( \mu \equiv \mu_1 + \frac{1}{2}\sigma^2 \). Substituting for consumption innovation in (2.2), the \( \gamma_t \)-process can be rewritten.
as:

\[
\frac{d\gamma_t}{\gamma_t} = \mu_{\gamma_t} dt + \sigma_{\gamma_t} dB_t + J_{\gamma_t} dN_t
\]  

(2.4)

where

\[
\begin{align*}
\mu_{\gamma_t} &= k \frac{\gamma_t - \gamma_t^-}{\gamma_t} + \alpha \frac{\gamma_t - \beta}{\gamma_t} \mu_b \\
\sigma_{\gamma_t} &= -\alpha \frac{\gamma_t - \beta}{\gamma_t} \sigma < 0 \\
J_{\gamma_t} &\equiv \frac{\gamma_t^-}{\gamma_t} - 1 = -\alpha \gamma_t^{-\beta} b > 0
\end{align*}
\]  

(2.5)

where \(\gamma_t^-\) denotes the value of \(\gamma\) an instant before the occurrence of jump. Since \(\gamma_t > \beta\), \(\text{sign}(\sigma_{\gamma_t}) = -\text{sign}(\sigma)\) and \(\text{sign}(J_{\gamma_t}) = -\text{sign}(e^b - 1)\), implying a negative consumption innovation, whether driven by diffusions or by jumps, leads to a positive innovation in \(\gamma_t\), or equivalently, a positive innovation in the agent’s risk aversion \(\psi_t\) and a negative innovation in the surplus ratio \(S_t\). In their absolute values, \(\sigma_{\gamma_t}\) and \(J_{\gamma_t}\) are both increasing in \(\gamma_t\), implying more volatilities of risk aversion when the agent becomes more risk averse.

With external habit formation, the pricing kernel \(\Lambda_t\) in the economy is given by

\[
\Lambda_t = e^{-\rho t} u_c(C_t, H_t) = e^{-\rho t} \frac{\gamma_t}{C_t}.
\]  

(2.6)

(2.6) implies that \(\Lambda_t\) is determined by both the aggregate consumption \(C_t\) denoting the economic fundamentals, and by the representative agent’s risk aversion through its monotone relation with \(\gamma_t\) which represents the market sentiment. As widely acknowledged, both economic fundamentals and market sentiment are important for asset pricing. By Ito’s lemma with jumps,

\[
\frac{d\Lambda_t}{\Lambda_t} = \mu_{\Lambda_t} dt + \sigma_{\Lambda_t} dB_t + J_{\Lambda_t} dN_t,
\]  

(2.7)

where

\[
\begin{align*}
\mu_{\Lambda_t} &= -\rho - \mu + \sigma^2 + \mu_{\gamma_t} - \sigma \gamma_t \\
\sigma_{\Lambda_t} &= \sigma_{\gamma_t} - \sigma \\
J_{\Lambda_t} &\equiv \frac{\Lambda_t^-}{\Lambda_t} - 1 = e^{-b} (J_{\gamma_t} + 1) - 1 > 0
\end{align*}
\]  

(2.8)

where \(\Lambda_t^-\) denotes the value of \(\Lambda\) an instant before the occurrences of jump. By combining (2.8) with (2.5), a negative consumption innovation, whether driven by diffusions or by jumps, leads to an amplified positive innovation in the pricing kernel

---

1The name "peso problem" is attributed to Milton Friedman’s comments about the effect of infrequent but disastrous events on the Mexico peso market in the early 1970s.
through the induced $\gamma_t$—innovations, which characterizes the agent’s overreaction to variations of economic fundamentals under habit formation. Consistent with the literature, we refer to $-\sigma_A > 0$ and $-J_A < 0$ as the price of the diffusive risk and the price of the jump risk, respectively, whose signs indicate that positive return volatility and negative price jumps will receive positive compensations. Like $|\sigma_{\gamma t}|$ and $|J_{\gamma t}|$, $|\sigma_A|$ and $|J_A|$ are both increasing in $\psi$, implying larger compensation per unit of risk for higher degree of risk aversion which coincides with bad economic states when consumption is close to the agent’s habit level.

**B. Modeling defaults**

In this paper, we assume that the investor has the same information set as the market – incomplete knowledge of the firm’s fundamental (e.g., Jarrow, Lando, Turnbull (1997), Duffie and Singleton (1999), Duffie and Lando (2001) and Giesecke (2006)). Given the scope of our study, this approach has the following advantages. First, it can be tractably embedded into the habit formation framework which are known to price well equity and index options. Second, bond cash flows at the event of default are based only on observable which are easy to calibrate. Third, this approach, as shown later in section IV.B, facilitates the direct study of default risks which can be conveniently compared to risk premiums implicit in other assets.

More specifically, we treat default as a separate pricing regime, and the occurrence of default is modeled as the switch from the no-default (the first) to the default (the second) regime. The distribution of default time is governed by the continuous time transition matrix $\Gamma$

$$
\Gamma = \begin{bmatrix}
-\lambda_1 & \lambda_{12} \\
\lambda_{21} & -\lambda_2
\end{bmatrix},
$$

where $\lambda_{ij}$ ($i \neq j$), which equals $\lambda_i$, denotes the switching intensity from regime $i$ to regime $j$. Since the bond payoff is irreversibly changed upon bankruptcy, we further impose that default regime is absorbing by setting $\lambda_{21} = \lambda_2 = 0$. Accounting for different credit classes, we rewrite

$$
\Gamma^j = \begin{bmatrix}
-\lambda_D^j & \lambda_D^j \\
0 & 0
\end{bmatrix}, \quad (2.9)
$$

where $\Gamma^j$ and $\lambda_D^j$ denote, respectively, the transition matrix and the default intensity.
for bonds in the $j$th credit class. For simplicity, we depress the superscript "$j$" in $\Gamma$ and $\lambda_D$ thereafter.

Our paper focuses on investment grade (IG) bonds whose pricing is hard to reconcile using structural models. In the general sense, defaults can be decomposed into the systematic components which are triggered by consumptions, with intensity denoted by $\lambda_D^{sys}$, and the idiosyncratic component, with intensity denoted by $\lambda_D^{idio}$, which are independent of consumption jumps. Mathematically,

$$\Gamma = \Gamma^{sys} + \Gamma^{idio}, \quad (2.10)$$

where

$$\begin{align*}
\Gamma^{sys} &\equiv \begin{bmatrix} -\lambda_D^{sys} & \lambda_D^{sys} \\ 0 & 0 \end{bmatrix}; \\
\Gamma^{idio} &\equiv \begin{bmatrix} -\lambda_D^{idio} & \lambda_D^{idio} \\ 0 & 0 \end{bmatrix}.
\end{align*} \quad (2.11)$$

We derive our theoretical valuations for defaultable bonds under the general (2.10)–(2.11).\footnote{Conditional on no default, $\Gamma^{sys}$ and $\Gamma^{idio}$ are equivalent to two Poisson processes $dN_D^{sys}$ and $dN_D^{idio}$ with intensities $\lambda_D^{sys}dt$ and $\lambda_D^{idio}dt$, respectively. It can be easily shown that the sum of two independent Poisson processes is still a process, which in this case models the total default.} Empirically, Bakshi, Madan, and Zhang (2006) find that defaults for IG bonds are usually related to severe economic conditions which are modeled by consumption jumps in our setup. Their finding is consistent with the observation that, on the annual basis, IG bonds have much lower probability of defaults than that of economic disasters (below 1% for five-year Aaa and Baa bonds vs 1.7% for consumption jumps). We recognize these empirical evidence in our numerical calculations. In Section VI, we provide the robustness checks on our results by varying the fraction of systematic defaults and idiosyncratic defaults relative to total defaults.

C. Stochastic default loss

Throughout the paper, we focus on zero-coupon defaultable bonds\footnote{Consistent with the literature (e.g., CCDG), we find that the implied credit spreads are extremely insensitive to the specification of coupon rate. Focusing on zero-coupon bonds is also consistent with the convention of studying zero-coupon default-free bonds.} which pays the unit face value at expiration date contingent on no default. To capture default risks, we need to account for bond losses due to defaults. The most natural pricing convention in practice seems to be the fractional recovery of face value (RFV) under which bondholders recover a fraction of the bond face value upon bankruptcy. There are
many empirical evidence that favor RFV assumption (see, among others, Guha and Sbuelz (2003) and Acharya et. al. (2006)). Therefore, we assume in this paper that bondholders recover $1 - L_t$ of the bond face value upon bankruptcy, where $L_t$ denotes the loss rate at the time of default.

To facilitate the exposition of model mechanism, we also consider another arrangement under which the asset makes the residual payment $1 - L_T$ only at its expiration date $T$, conditional on default prior to $T$. In the following, we refer to the second arrangement, the terminal payment (TP) case. If the loss rate is constant, this assumption boils down to what is commonly known in the literature as recovery of Treasury (RT). Another reason for considering the latter case is the recent work of Bakshi, Madan and Zhang (2009) who provide supports that the recovery of Treasury approach seems to allow for better econometric fits for defaultable bonds. We derive closed form solutions for defaultable bonds under both RFV and TP arrangements.

There are ample studies in the literature suggesting that default loss is time varying (e.g., Altman and Kishore (1996)). In addition, asset sales of distressed firms suffer from large discounts if the entire industry or the economy experiences liquidity constraints (Shleifer and Vishney (1992) and Acharya et al (2009)) implying a countercyclical loss rate. In view of these evidence, we model the loss given default as a stochastic mean reverting process which is countercyclical.

More specifically, we impose that $L_t$ is perfectly correlated with innovations in log consumption as follows:

$$dL_t = k_L(\bar{L} - L_t)dt - \alpha_L(dC_t - E_t[dC_t]), \quad (2.12)$$

where $\bar{L}$ is the long run average of the recovery rate; $k_L$ controls the speed of mean reversion; $\alpha_L > 0$ captures the sensitivity of $L_t$ to consumption innovations. Intuitively, negative consumption innovation implies a worsening of macroeconomic condition which tends to raise the loss rate\(^4\). Using consumption process, we write (2.12) more explicitly as follows:

$$dL_t = k_L(\bar{L} - L_t)dt - \alpha_L \sigma dB_t - \alpha_L b(dN_t - \lambda dt). \quad (2.13)$$

Like the $\gamma$-process, $L_t$ jumps upward following the strike of economic disasters. The

\(^4\)In Appendix A, we extend this specification of loss rate to include an idiosyncratic shock in addition to consumption shock. Closed form solution are also developed in this setup.
difference is that we assume constant jump size for \( L_t \) whereas \( \gamma \)-jump size depends on its value an instant before the disaster.

## III Equilibrium Asset Prices

Given the common pricing kernel and the cash flow processes characterizing different asset classes, we derive in this section their closed-form valuations. We first start by deriving the expression of default-free bonds. Second, we move to the pricing of defaultable bonds. We focus on the RFV assumption case and refer the reader to Appendix A for the expressions under the TP arrangement. Lastly, we obtain the formulas for aggregate equity and its derivative.

### A. Default-free bonds

**Proposition 1** Denote by \( P_{t,\tau}^0 \) the price of a zero-coupon default-free bond with \( \tau \)-period to expiration. Then (a)

\[
P_{t,\tau}^0 = \alpha_1^0(\tau) + \alpha_2^0(\tau) \frac{1}{\gamma_t},
\]

where \( \alpha_1^b,0(\tau) \) and \( \alpha_2^b,0(\tau) \) are all positive and given by (A.54)–(A.55) in Appendix A.

(b) The instantaneous return of default-free bond follows:

\[
\frac{dP_{t,\tau}^{b,0}}{P_{t,\tau}^{b,0}} = \mu_{P_t} dt + \sigma_{P_t} dB_t + J_{P_t} dN_t,
\]

where

\[
\mu_{P_t}^{b,0} = \alpha_2^0(\tau) \frac{1}{\gamma_t} \frac{1}{P_{t,\tau}^{b,0}} \sigma_{\gamma},
\]

\[
J_{P_t}^{b,0} = \frac{P_{t,\tau}^{b,0}}{P_{t,\tau}^{b,0}} - 1 = \alpha_2^0(\tau) \frac{1}{\gamma_t} \frac{1}{P_{t,\tau}^{b,0}} \left( \frac{1}{1 + J_\gamma} - 1 \right),
\]

where \( t^- \) denotes the time an instant before the bond price jump; \( \mu_\gamma \), \( \sigma_\gamma \), and \( J_\gamma \) are, respectively, the drift, the diffusion, and the jump size of \( \gamma_t \) whose expressions are in (2.5).

**Proof.** The closed form for \( P_{t,\tau}^0 \) can be derived as the special single-regime case of defaultable bond pricing when the default state is shut down. Expressions for \( \mu_{P_t}^0 \) (omitted), \( \sigma_{P_t}^0 \), and \( J_{P_t}^0 \) follow Ito’s lemma with jumps (e.g., Appendix F of Duffie, 2001). ■
Eq. (3.1) shows that $P^0_{t,t}$ loads negatively on risk aversion $\gamma_t$ and hence positively on the surplus $S_t$. By combining the bond return process (3.2) with the pricing kernel process (2.7), the implied bond risk premium is

$$BP^0_t = -\sigma_{\Lambda t}^0 \sigma_{P_t}^0 - \lambda J_{\Lambda t} J_{P_t}^0,$$

(3.4)

where $\sigma_{\Lambda t}$ and $J_{\Lambda}$ are the diffusive volatility and the jump size of the pricing kernel given by (2.8). In (3.4), the two terms are compensations for diffusive and jump risks, respectively. As discussed also in Wachter (2006), real (default-free) bonds are risky under habit formation. Their yields are thus low for high consumption surplus which generates the upward-sloping yield curve. This relation implies a negative covariance between dynamics of bond returns and the pricing kernel, and hence a positive bond risk premium.

It is straightforward to verify that in the limit case when $\tau \to 0$, the implied bond yield $P^0_{t,t} = -\frac{1}{\tau} \ln P_{t,t}^0$ approaches the short term interest rate $r$ which by definition equals

$$r = -E \left[ \frac{\Lambda_t}{\Lambda_{t+t}} \right] = -\mu_{\Lambda} - \lambda J_{\Lambda},$$

where $\mu_{\Lambda}$ and $J_{\Lambda}$ are the drift and the jump size of the pricing kernel given in (2.8).

**B. Defaultable bonds**

Let $P^{b,RFV}_{t,t}$ be the prices of a zero-coupon defaultable bond with $\tau-$period to expiration, where the superscript denotes recovery of face value. More specifically,

$$P^{b,RFV}_{t,t} = \left[ P^{b,RFV}_{t,t} (1), P^{b,RFV}_{t,t} (2) \right]' \in \mathbb{R}^2,$$

where $P^{b,RFV}_{t,t} (1)$ and $P^{b,RFV}_{t,t} (2)$ denote, respectively, the defaultable bond prices conditional on no-default (the first) and the default (the second) regime.

**Proposition 2** Under RFV arrangement,

$$P^{b,RFV}_{t,t} = \alpha_{1}^{b,RFV} (\tau) + \alpha_{2}^{b,RFV} (\tau) \frac{1}{\gamma_t} + \alpha_{3}^{b,RFV} (\tau) L_{t} + \alpha_{4}^{b,RFV} (\tau) \frac{L_{t}}{\gamma_t},$$

(3.5)

where $\alpha_{j}^{b,RFV} (\tau) \in \mathbb{R}^2$ for $j = 1, 2, 3, 4$; their expressions are given by (A.42)-(A.45) in Appendix A.
Proof. (3.5) follows a three-factor application of the linear factor processes augmented with structural changes (LS) proposed by Du and Elkamhi (2011). The accuracy of the closed form valuation is verified by simulation for various \( \tau \).

To simplify notation, we write thereafter:

\[
P_{t,\tau}^{RFV} \equiv P_{t,\tau}^{b,RFV} (1) \; ; \; \alpha_j^{RFV} (\tau) \equiv \alpha_j^{b,RFV} (\tau, 1) \; (j = 1, 2, 3, 4).
\]

From (3.5),

\[
P_{t,\tau}^{RFV} = \alpha_1^{RFV} (\tau) + \alpha_2^{RFV} (\tau) \frac{1}{\gamma_t} + \alpha_3^{RFV} (\tau) L_t + \alpha_4^{RFV} (\tau) \frac{L_t}{\gamma_t}, \tag{3.6}
\]

which gives the closed form valuation conditional on no default. It is easy to check numerically that \( \alpha_1^{RFV} (\tau), \alpha_2^{RFV} (\tau), \) and \( \alpha_4^{RFV} (\tau) \) are positive, while \( \alpha_3^{RFV} (\tau) \) and \( \alpha_4^{RFV} (\tau) \frac{1}{\gamma_t} \) are negative.\(^5\) Therefore, (3.5) says that a high risk aversion and a high loss rate both translate into a low defaultable bond price which is quite intuitive. Appendix C gives detailed analysis about the four pricing coefficients in terms of their separate impacts on the implied credit spreads.

Next, we use a separate proposition to summarize the dynamics of \( P_{t,\tau}^{RFV} \).

**Proposition 3** The instantaneous return of RFV defaultable bond conditional on no default follows:

\[
\frac{dP_{t,\tau}^{RFV}}{P_{t,\tau}^{RFV}} = \mu_P^{RFV} dt + \sigma_P^{RFV} dB_t + J_{P_t}^{RFV} dN_t^D + \left[ \frac{1}{P_{t,\tau}^{RFV}} - 1 \right] dN_{D_t}, \tag{3.7}
\]

where \( dN_{D_t} \) is a Poisson process with intensity \( \lambda_d dt \) which models the arrival of default; \( dN_t^D \) is another Process independent of \( dN_t^D \) which models the arrivals of the economic disasters that do not induce default for the given credit class;

\[
\sigma_{P_t}^{RFV} = -\alpha_2^{RFV} (\tau) \frac{1}{\gamma_t} P_{t,\tau}^{RFV} \sigma_\gamma - \alpha_3^{RFV} (\tau) \frac{1}{\gamma_t} P_{t,\tau}^{RFV} \alpha_4^{RFV} (\tau) \frac{1}{\gamma_t} P_{t,\tau}^{RFV} \left( L_t \sigma_\gamma + \alpha_L \sigma \right)
\]

\[
J_{P_t}^{RFV} \equiv \frac{P_{t,\tau}^{RFV}}{P_{t,\tau}^{RFV}} - 1 = \alpha_2^{RFV} (\tau) \frac{1}{\gamma_t} P_{t,\tau}^{RFV} \left( \frac{1}{1+J_\gamma} - 1 \right) - \alpha_3^{RFV} (\tau) \frac{1}{P_{t,\tau}^{RFV}} \alpha_L b + \alpha_4^{RFV} (\tau) \frac{1}{\gamma_t} P_{t,\tau}^{RFV} \left[ L_t - \left( \frac{1}{1+J_\gamma} - 1 \right) - \alpha_L b \frac{1}{1+J_\gamma} \right]. \tag{3.8}
\]

\(^5\)Note from (2.2) that \( \gamma_t \) has the lower bound \( \beta \) which curbs the magnitude of the positive term, \( \alpha_4^{RFV} (\tau) \frac{1}{\gamma_t} \). Thus, the \( \alpha_3^{RFV} (\tau) \) term always dominates for \( \alpha_3^{RFV} (\tau) + \alpha_4^{RFV} (\tau) \frac{1}{\gamma_t} \) to be negative.
where \( \mu_\gamma, \sigma_\gamma, \) and \( J_\gamma \) are, respectively, the drift, the diffusion, and the jump size of \( \gamma_t \) whose expressions are in (2.5).

**Proof.** In (3.7), the orthogonal decomposition of \( dN^\perp_t \) and \( dN^D_t \) takes into account that bond price jump size not accompanied with default, which is denoted by \( J^{RFV}_{Pt} \), differs from the jump size triggered by default which equals \( \frac{1-L_t}{P^c_t} - 1 \). In the latter case, we’ve used that under RFV arrangement, bond price upon default immediately jumps to recovery rate \( 1 - L_t \). Finally, an application of Ito’s lemma with jump for (3.6) gives (3.8).

From discussions in Section II.B, bond defaults in general can be decomposed into a systematic component modeled by \( dN^{sys}_{Dt} \) which is associated with consumption jump, and an idiosyncratic component modeled by \( N^{sys}_{Dt} \) which is independent of consumption jumps. On the other hand, for the give credit class, consumption jump can be decomposed into a component that is associated with the default modeled by \( dN^{sys}_{Dt} \), and a component that is independent of default which is modeled by \( dN^\perp_t \). In other words,

\[
dN_t = dN^{sys}_{Dt} + dN^\perp_t,
\]

where \( dN_t \) models the arrival of disaster with intensity \( \lambda \). It is worth noting that the above decomposition is not unique and it varies for different credit classes. Combining (3.7) with (2.7), we obtain the following risk premium demanded by holding the RFV bonds:

\[
BP_t = -\sigma_M \sigma^{RFV}_{Pt} - \lambda^\perp J_M J^{RFV}_{Pt} - \lambda^D J_M \left[ \frac{1-L_t}{P^RFV_t} - 1 \right],
\]

In (3.10), the three terms are compensations for diffusive risks, jump risks (not accompanied with default), and default risks, respectively. Bond risk premiums are closely related to the implied credit spreads, and we leave the details to Section IV.B.

**C. Aggregate equity and its derivatives**

**Proposition 4** Equating aggregate dividend with aggregate consumption, and denote by \( P^s_t \) the aggregate dividend. Then (a) the price-consumption ratio is given by

\[
P^s_t = \left( \frac{1}{\rho + k} + \frac{k \gamma}{\rho (\rho + k) \gamma_t} \right) C_t.
\]

16
(b) the instantaneous equity return follows:

$$\frac{dP_t^s}{P_t^s} = \mu_{P_t}^s dt + \sigma_{P_t}^s dB_t + J_{P_t}^s dN_t,$$

(3.12)

where

$$J_{P_t}^s \equiv \frac{P_t^s}{P_{t-}^s} - 1 = e^b \left[ 1 + \frac{k\gamma}{\rho(\rho+k)} \gamma_{t-} - \frac{1}{\gamma_{t-}} \left( \frac{1}{1+J_{t-}} - 1 \right) \right] - 1,$$

(3.13)

where $X_t \equiv \frac{P_t^s}{C_t}$.

The above proposition is proved in Appendix A of Du (2011). Like that of defaultable bond, valuation of equity is also depressed (relative to consumption) when investors become more risk averse. Using (3.13), the equity premium and the total equity return volatility are

$$EP_t = -\sigma_M \sigma_{P_t}^s - \lambda J_{P_t}^s,$$

(3.14)

$$\text{vol}R_t = \sqrt{\left( \sigma_{P_t}^s \right)^2 + \lambda (J_{P_t}^s)^2},$$

(3.15)

where the two terms in (3.14) and (3.15) are due to diffusions and price jumps, respectively.

At period $t$, the equilibrium price of a put option written on the aggregate equity is by definition:

$$O_t = E_t \left[ \frac{\Lambda_{t+\tau}}{\Lambda_t} \max(K - P_{t+\tau}^s, 0) \right],$$

where $K$ denotes the the strike price; $\tau$ denotes the time to maturity. Unlike equity price, option prices cannot be derived in closed form. we therefore simulate a large number of the $O_t$—realizations and use their average as the option price implied from the model. Given $O_t$, the implied Black-Scholes volatility (B/S-vol) is computed as

$$\text{B/S-vol}_t = BSC^{-1}(\tau, K, O_t, r_{t,t+\tau}, CP_{t,t+\tau}),$$

where $BSC^{-1}$ is the inverse of the Black-Scholes formula for the put option, inverted over the argument $\sigma$; $r_{t,t+\tau}$ and $CP_{t,t+\tau}$ are the interest rate and the dividend-price ratio over the period of $[t, t + \tau]$. Following the convention, we quote option prices in terms of B/S-vol in the following analysis. To use the common metrics for comparing
option prices, we fix \( r_{t,t+\tau} \) and \( CP_{t,t+\tau} \) at 5% and 3% throughout the paper\(^6\).

IV Model mechanism

This section provides a discussion on model mechanism. We first calibrate the model on which quantitative calculations are based. Second, we analyze the importance of relating investment grade defaults to economic states versus independent default assumption. Third, we perform an analysis on the impact of countercyclical loss and time varying risk aversion on the required premium for 4-years Baa-Treasury spreads.

A. Model calibration

The parameters not related to economic disasters are reported in Panel A of Table I. For the calibration of consumption and habit formation process, we follow the calibration in Du (2011) and the readers are referred to that paper for detailed discussions on the data selection, identification issues and steps of the calibration. For the two parameters related to default loss rate process, we set \( k_L = 0.4 \) implying modest degree of mean reversion, and we set \( \alpha_L = 2 \), much less than \( \alpha \), implying that \( L \) is much less sensitive to consumption innovations compared to that of \( \gamma \). This calibration are in line with Doshi (2011) who estimates a stochastic recovery process from a set of senior and subordinated CDS contracts. While (2.12) does not guarantee that \( L \) stays within zero and one, numerically we find that with fairly low \( \alpha_L = 2 \) and for 100,000 simulated paths for \( L \) starting from \( L \), \( L \) never turns negative, and with only 0.05% rises above one. Hence, it seems appropriate to interpret \( L \) as the loss rate with the bond face value normalized to one.

Panel B of the same table reports variables related to disasters and defaults. Economic jumps strike once every 45 years with 17.2% jump size in log consumption, or equivalently, a 15.8% jump size in consumption itself measured by \( e^b - 1 \). This number controls the price of both jump and default risks, and it is estimated using a panel of S&P 500 index option data by Du (2011).\(^7\)

---

\(^6\)In the model, both \( r \) and \( CP \) are time varying. Since our analysis focuses on comparing the option prices implied from the model vs. from the data, the results are unaffected so long as the common \( r \) and \( CP \) are used.

\(^7\)While still looking very large compared to the maximum 9.9% annual consumption contraction during the Great Depression, it is much more in line with the empirical observations than those implied from peso problem models with standard CRRA preference (e.g., Barro, 2006 and Wachter
Following CCDG and consistent with empirical data available in Moody’s manuals, we set the average loss rate $L$ to 0.551. We also set the four-year cumulative default probability for Aaa and Baa bonds to 0.04% and 1.55%, respectively. The annualized default intensity for these two bonds are thus $\nu^{D,Aaa} = 0.01\%$ and $\nu^{D,Baa} = 0.3875\%$.

Unlike bond and equity prices, option prices cannot be derived in closed-form. We thus apply interpolation to efficiently compute the unconditional option pricing moments implied from the model. First, we obtain $\{S_i\}_{i=1}^N$, a large number of the realizations of the surplus ratio realizations $S$ in its stationary distribution region. Next, we simulate option prices conditioned on $n$ interpolation nodes within the range of $\{S_i\}_{i=1}^N$. Finally, we use the $n$ conditional prices to interpolate the prices conditioned on each of realizations in $\{S_i\}_{i=1}^N$, whose average is reported as the unconditional option moments implied from the model. Experiments show that Chebyshev interpolation with 10 interpolation nodes provides fairly accurate results compared to direct simulations.

**B. Disaster-triggered defaults**

We start our analysis by exploring the structure of the risk premia demanded to hold defaultable and default free bonds. One advantage for pricing assets in closed form is to allow as straightforward comparison of premium and analyze how jumps and diffusions influence these assets. First, we start by comparing bond risk premium $BP_t$ for defaultable bonds and its counterpart $BP^0_t$ for default-free bonds. To highlight the importance of relating default to the states of the economy, we also derive in Appendix A.3.2 the implied bond risk premium, denoted by $BP_{t,ind}^t$, when defaults are independent of economic disasters. To facilitate comparison, we rewrite formulas for $BP_t$, $BP_{t,ind}^t$, and $BP^0_t$ as follows:

$$BP_t = -\sigma_M \sigma_{P_t}^{\beta,IP} - \lambda^1 J_M J_{P_t}^{\beta,IP} - \lambda^D J_M \left[1 - \frac{L_t}{P_t^{\beta,IP}} - 1\right], \quad (4.1)$$

$$BP_{t,ind}^t = -\sigma_M \sigma_{P_t}^{\beta,ind} - \lambda J_M J_{P_t}^{\beta,ind}, \quad (4.2)$$

$$BP^0_t = -\sigma_M \sigma_{P_t}^{\beta,0} - \lambda J_M J_{P_t}^{\beta,0}, \quad (4.3)$$

2006 requires a size of 30%)
where related variables are defined in (3.3), (3.8) and (A.58). The first observation is that the three premiums share the first two terms which are due to diffusive risks and jump risks, respectively. Compared to $BP_t^{ind}$ and $BP_t^0$, $BP_t$ has an additional term, $-\lambda^D J_M \left[ \frac{1 - L_t}{L^m} - 1 \right]$, which compensates for default risks. Note we differentiate default risks from jump risks in our model: while both are attributed to bond price jumps, only the former are driven by defaults.

To provide a clear understanding about relating defaults of IG bonds to severe economic states, we plot results about default-free, independent defaults, and disaster-triggered defaults in response to changes of the surplus ratio $S$ in Figure 1, where we keep the loss $L_t$ at its empirical average $\bar{L}$. The top left panel plots together the three jump risk premiums. While premiums from default-free and independent defaults are virtually indistinguishable, the premium from disaster-triggered default is in general higher. The reason is that bond prices and hence their returns are largely unaffected by default losses if defaults are not priced. On the other hand, bond returns become more volatile implying higher quantity of risks when defaults are directly priced.\(^8\) The pattern for the comparison of the three diffusive risk premiums are very similar, and we do not report the figure to save space.

The top right panel of Figure 1 plots the default risk premiums. Naturally, default-free bonds do not claim default risk premium. Default risk does not influence $BP_t^{ind}$ either since defaults independent of economic shocks are not priced. In terms of magnitude, default risk premium in $BP_t$ is lower than the jump risk premiums, which is mainly due to the fact that default intensity for investment grade bonds is much smaller than disaster intensity. The likelihood of consumption jumps is estimated to be 1.7% while it is 1.4% and 0.5% for Baa and Aaa, respectively.

The bottom two panels plot together the premium differences, $BP_t^{ind} - BP_t^0$ and the implied Baa-Treasury spreads under both independent default (left) and disaster-

\(^8\)Note jump risk premiums in all three models are countercyclical only for relatively large $S$, and they become procyclical for low values of $S$ (this is also true for $BP_t^{ind}$). First, given that in our calibration $S$ is around 0.03, we view very low values of $S$ as extreme recessionary times. But still, this procyclicality can look intriguing at first sight. As explained also in MSV, the reason lies in the functional form of external habit models. The volatility of $S$ must vanish as $S \to 0$ to prevent negative marginal utility. This feature of habit models results in a decrease in both diffusive bond return volatility and bond price jump size conditional on no default, hence the decrease of the corresponding components in bond risk premiums. To put it differently, the model interestingly makes the difference between total risk and systematic uncertainty. As risk becomes eminent the price of risk per unit of risk increases but the quantity of systematic risk in fact converges to zero. The second effect dominates and lower the demanded premium.
triggered default (right). As shown in the bottom left panel, premium differences stay close to zeros when defaults are not priced. With virtually no extra premium claimed by defaultable bonds, credit spreads only reflect the expected value of default loss when investors are risk neutral. The implied level (around 21 bp) falls far below its empirical counterpart, usually over 140 bp. In addition, the implied spreads are essentially constant, just like the premium difference, which is inconsistent with the countercyclical credit spreads observed in the data. These findings show that a model that does not tie defaults to economic states would face difficulties at generating observed spreads for IG bonds.

By comparison, differences between $BP_t$ and $BP_t^0$ are much larger than $BP_t^{ind} - BP_t^0$. As a result, the implied credit spreads are also much higher, well above 100bp for low $S$. Unlike default risk premium, credit spread is countercyclical and hence monotonically decreasing in $S$. Intuitively, investors require a large spread to hold defaultable bonds when defaults coincide with bad states of the economy, and this is particularly the case when their risk aversion is high for low $S$. To put it differently, credit spreads contain both an expected loss and risk premium components. The expected loss component is monotonically related to loss rate which rises as the states of the economy worsens. However, as explained in footnote 12, the quantity of systematic risk decreases as $S$ goes to zero leading to a reduction in risk premium. The first channel dominates for low level of $S$ which ensures the intuitive and desirable result of countercyclical spread.

The impact of disaster-triggered defaults on the premium difference and countercyclical and sizable spreads, can also be understood from the implied strong positive covariance between the pricing kernel and the default time. The pricing kernel is high and default is more likely upon the strike of consumption jumps. As emphasized by CCDG, this covariance structure is one of the two channels to generate the high credit spread. Another channel in our setup (which is not considered yet in Figure 1) lies in the strong positive covariance between the pricing kernel and the loss rate. This latter channel would play a stronger effect at generating lower defaultable bond prices in addition to increasing covariance with the pricing kernel. We return to this point in details below where we also describe our different mechanism from CCDG.
C. Time varying risk aversion and default losses

As recognized in the equity literature, one reason that habit models are able to explain the equity premium and its volatility lies in the time variation of the representative agent’s risk aversion. In addition, our setup allows for countercyclical loss rate which is a well documented empirical fact in the credit literature. In this section, we carry on comparative statics to gauge the importance of both components on bond premiums and total spreads. Again, we do this exercise for 4-year Baa bonds.

To evaluate the importance of countercyclical \( \gamma \) on risk premia, we set \( k = \alpha = 0 \), and plot in top panels of Figure 2 bond risk premiums and their components under both the full model and the special case with constant \( \gamma \) fixed at \( \bar{\gamma} \). The top left panel plots together the diffusive risk premium with and without variations in \( \gamma \). To understand the large difference of the diffusive premium, note that when \( \gamma \) stops to be stochastic, \(-\sigma_A\) which serves the price of diffusive risk degenerates to consumption diffusive volatility \( \sigma \) which is very low in comparison. The top right panel Figure 2 plots together the sum of the jump risk premium and the default risk premium in the same two cases. Since defaults are triggered by consumption jumps in our model, the price of jump risk and the price of default risk are both given by \(-J_A\). When \( \gamma \) stops to be stochastic, \(-J_A\) degenerates to \(- (e^{-b} - 1)\) which is also much lower compared to the stochastic \( \gamma \) case.

As previously shown in Figure 1, the expected loss component of IG bond is small and credit spreads becomes closely related to bond risk premiums. Thus, the implied credit spread with constant \( \gamma \) is also much lower. Indeed, the bottom left panel of the same figure illustrates that while Baa-Treasury spread varies from 90–190 bps in the full model, it only takes values between 26–53 bps in the special model with constant \( \gamma \). In sum, even after allowing bond cash flows to be correlated with the states of the economy via a countercyclical loss rate, time-varying attitudes toward risks do play an important role in generating higher credit spread.

We now turn our attention to the impact of countercyclical default loss. From (2.12), negative consumption innovation which tends to occur during recessions drives up the loss rate \( L \), and hence drives down the recovery rate. Compared to the case with a constant \( L \), default becomes riskier since debt-holders incur larger losses exactly when their marginal utility is high. To put it into another perspective, countercyclical \( L \) generates strong positive covariance between the pricing kernel and the loss rate, since both increase upon negative consumption innovations. This is particularly
the case upon downward consumption jumps which trigger upward jumps in both the pricing kernel and the loss rate, hence the further depression of the valuation of defaultable bonds. Indeed, shutting down this channel by setting \( k_L = \alpha_L = 0 \) substantially decreases the implied Baa-Treasury spread. As illustrated in the bottom right panel of Figure 3, while the spread varies from 73–348 bps in the full model, it only takes values between 40–130 bps in the special model with constant \( L \).

In what follow we shed light on the importance of both countercyclical \( L \) and \( \gamma \), and explicitly illustrate the differences of our model mechanism with respect to CCDG which is closely related to the present paper. Our first difference with their setup is that we consider a peso component in consumption. This addition turns out to be crucial in pricing not only defaultable bonds but also aggregate equity and its derivative – a point that will be explicitly covered in section V.C. Since CCDG’s model is not designed to price options, we focus on differences with respect to defaultable bonds. For this purpose, we borrow from the excellent representation in CCDG and write the price of a zero coupon bond \( P \) as:

\[
P = E [\Lambda(A_t, t)(1 - 1_{\tau \leq T}(A, B, t) \times L_r(A, B, t))]
\]

where \( \Lambda \) represents the pricing kernel; \( A \) denotes a vector of macro-economic variables such as \( \gamma \) in our setup; \( B \) denotes the vector of firm specific (idiosyncratic) variables. The differentiation between \( A \) and \( B \) is consistent with the decomposition of default transition matrix given by (2.11).

Dropping the time \( t \) and expanding the above formula as in CCDG, we get

\[
P = E [\Lambda(A_t)] * E [1 - 1_{\tau \leq T}(A, B) \times L_r(A, B)] - cov (\Lambda(A_t), 1_{\tau \leq T}(A, B) \times L_r(A, B)).
\]

(4.4)

Huang and Huang (2003, HH) emphasize that both \( E [1_{\tau \leq T}(A, B)] \) and \( E [L_r(A, B)] \) should be fitted to their empirical levels when testing a model. In order to generate a low \( P \) and hence the high credit spread. CCDG advocate increasing the covariance term on the RHS of (4.4) by a countercyclical default boundary or a countercyclical Sharpe ratio. In addition to the fact that it is unclear how these additions would help in pricing index options, \( L_r(A, B) \) in their setup is not a function of macro condition. Instead, \( L \) is set to be constant or can be viewed as \( L_r(B) \). This raises an important question: do we still require countercyclical risk aversion to capture credit spread
if we allow the loss rate to be countercyclical? Our framework provides an answer: in Figure 2, it is clear that \( \gamma \) and \( L \) both play complementary and reinforcing role at increasing the premium compensation for defaultable bonds. Therefore, while we agree with their comments that allowing for countercyclical \( L \) is not enough in the absence of countercyclical default rates, we emphasize that both are necessary to match observed spreads.

There is another important channel in our setup in addition to what is suggested by CCDG. To see it, rewrite (4.4) by

\[
P = E[\Lambda(A_t)] \ast (1 - E[1_{\tau \leq T}(A, B) \ast L_\tau(A, B)]) - cov(\Lambda(A_t), 1_{\tau \leq T}(A, B) \ast L_\tau(A, B)),
\]

and call the first term on the RHS the expectation term. In the absence of correlation between loss rates and defaults as in CCDG, \( E[1_{\tau \leq T}(A, B) \ast L_\tau(A, B)] = E[1_{\tau \leq T}(A, B)] \ast E[L_\tau(A, B)] \). However, this is not a viable restriction. In fact, it is well documented that default rates and loss rates are strongly and positively correlated. In our setup, this is ensured because both the default time \( \tau \) and the loss rate \( L_t \) are countercyclical and driven by consumption innovation. An application of Jensen lemma gives that:

\[
E[1_{\tau \leq T}(A, B) \ast L_\tau(A, B)] > E[1_{\tau \leq T}(A, B)] \ast E[L_\tau(A, B)] .
\]

In summary, allowing the loss rate to depend on macro conditions not only increases the covariance term as explained above, but also decreases the expectation term. We thus conclude that accounting for countercyclical and stochastic loss rate that is correlated with default rate plays a double good effect at decreasing bond prices to match the data.

### V Quantitative results

This section presents the model fit. We start by describing model implications about 5-year Corporate-Treasury spreads. These results are compared to CDS spreads in the data which are a cleaner proxy for default risk. Specifically, we perform our analysis by carrying on a decomposition of spreads into multiple sub-models which further illustrates the model mechanism. Next, we turn to implications about Baa-
Aaa spreads whose data values are available back to 1919. In this subsection, we present the performance of our model in explaining both the whole term structure of spreads as well as its time series. In Section V.C., we present an analysis of the common components in our modeling that help relate the equity, bonds and option markets.


In Section IV.C, we illustrate the impacts of $\gamma$ and $L$ on required risk premia for defaultable securities. Now we perform a further decomposition of our full model to allow an understanding of all the components of our framework. We focus on both 5-years Baa-Treasury and Aaa-Treasury spreads where the cumulative default rates are from Moody’s handbook.

We perform this exercise by considering the following four cases. In model 1, we assume independent defaults and set $k = \alpha = k_L = \alpha_L = 0$. In this model, bond risk premium is zero, and credit spread is only attributed to the default loss at the average level $\bar{L}$. In model 2, we allow disaster-triggered defaults but still set $k = \alpha = k_L = \alpha_L = 0$. In this model, channel 1, i.e., the positive correlation between pricing kernel and default time, is open, but we shut down variations in both states. In model 3, we allow disaster-triggered defaults and only allows variations in $L$ by setting $k = \alpha = 0$. In model 4, we allow disaster-triggered defaults and only allows variations in $\gamma$ by setting $k_L = \alpha_L = 0$. Model 5 is the full model. Figure 3 plots the implied Baa-Treasury and Aaa-Treasury spreads under each of the five cases. We start describing our results for Baa-Treasury spread first, and then we move to the Aaa-Treasury spread.

The top left panel illustrates spreads as a function of $S$ while default loss is kept at $L$. Credit spread attributed to default loss alone is tiny, around 29 bps.9 Adding channel 1 but shutting down state variations only increases the spreads marginally. One can think of no state variation case as the curve that one would get if using one of the risk neutral structural models calibrated in the Huang and Huang (2003).

---

9When pricing kernel is not correlated with defaults, defaultable bond price can be roughly computed as $P_t^b = E_t \left( \frac{X_t}{\hat{X}_t} \right) E_t \left( 1 - 1_{[\tau, T]} \bar{L} \right) = P_{t, 0}^b \left( 1 - \pi_{[t, T]} \bar{L} \right)$, where $P_{t, 0}^b$ denotes the default-free bond price; $\pi_{[t, T]}$ denotes the accumulative default probability from $t$ to $T$. The implied credit spread is thus $\frac{1}{T-t} \ln \left( 1 - \pi_{[t, T]} \bar{L} \right)$ which is around 30 bps for five-year Baa bonds.
Adding channel 2 by allowing for variation in default loss (model 3) almost doubles the implied credit spreads. The loss rates influence the covariance with the pricing kernel. However, the countercyclical variations in the loss rates in our calibration are smaller in comparison with the countercyclical variation in $\gamma$ in habit models. This effect reduces the countercyclically of spread when $\gamma$ is not stochastic. Allowing for habit formation (model 4) also adds substantially to the credit spreads. Finally, the full model further drives up the implied spreads to levels close to their empirical values for the 5-year CDS spread written on Baa firms.

The bottom left panel of Figure 3 illustrates the same decomposition for Aaa rated bonds. The same pattern reveals itself for different models. Note that in our full model, spread is countercyclical and also reaches higher levels, from 7 to 28 bp. This is encouraging given the average CDS spread for Aaa rated firms is around 10bp with a standard deviation of around 5bp (excluding 2007-2008 liquidity crisis).

Next, we turn our attention to the mechanism of our model with respect to different levels of loss rates. The top and bottom right panel of Figure 3 plots both 4 years Baa-Treasury and Aaa-Treasury spread respectively as a function of $L$ while risk aversion is fixed at $\gamma$. The patterns are similar to the left panels in that higher loss rate and higher risk aversion, i.e., lower surplus ratio, both drive up the implied credit spreads. An important difference is that model-implied credit spread can go negative for very low default loss if we shut down the countercyclical $L$. Appendix B provides interpretation for this very interesting finding which emphasizes the impact of real interest rate term structure on credit spreads. It is worth noting that our full model does not suffer from this counter-intuitive pattern which provide support that both countercyclical $\gamma$ and $L$ are needed.

Now, we compare our full model to the observed data on 5-year Baa-Treasury and Aaa-Treasury. On average, the model generates 181bp for Baa-Treasury and 10.3bp for Aaa-Treasury. These numbers are very close to CDS spreads on Baa and Aaa rated firms (153bp on average and 10 bp, respectively), and they fall below observed corporate bond spreads. The latter discrepancy lies in that our model only accounts for credit risk, and we do not consider other factors that affect corporate bond prices, such as taxes, call/put/conversion options and the lack of liquidity in the corporate bond markets. Several papers (Elton et al. (2001), Driessen (2005), Longstaff et al (2005), Covitz and Dowing (2007) and Chen, Lesmond and Wei (2007)) have investigated the decomposition of spreads into various components and show that the
aforementioned factors do have a sizable component. In light of these findings, we interpret our match for CDS averages and the underestimation for IG bonds as the strength of our model.

To provide comparison with the existing literature, long run risk models attribute a sizable component to default risk for medium and long term bonds. For example, Chen (2011) reports a model-implied spread of 45bp for Aaa while CDS spreads for similar rating is around 10 bp and corporate spread is around 53bp. Alternatively, using only habit formation, CCDG report a four-year Aaa spread of 1bp. As explained in the Introduction, the reason partly lies in the steeply upward-sloping yield curve in habit formation and the downward sloping yield curve in long run risk. Yield term structure in our setup is only slightly upward sloping, thanks to the peso component, which enables our model to obtain the match.

B. Baa-Aaa spread

To continue comparing our results to corporate bonds data, we restrict our attention in what follows to Baa-Aaa spreads. Our reasoning (and similar to a common practice in the literature) is that if the component of the credit spread due to these non-default ingredients is of similar magnitude for Aaa and Baa bonds, then their spreads, denoted by $SP_{BA}$, should be mostly due to credit risk. We recognize that the call feature on Baa bonds may be more valuable than the call feature on Aaa bonds. Also we know from Ericsson and Renault (2001) and He and Xiong (2011) that liquidity and credit are related. However, we think that relative differential sizes for IG bonds are small and can be ignored for the following comparison exercises.

B.1. Mean and standard deviation

Panel A of Table II reports the 4-year Baa-Aaa spreads under the five different models discussed in Section V.A. State-dependences are shut down in the first two models whose difference lies in whether corporate defaults are independent of (model 1) or coincide with (model 2) economic large downturns. The implied standard deviation of spreads are thus zeros in both models and the average spreads in the two models are also close to each other at around 20 bp. One can refer to Huang and Huang (2003) to notice similar low values generated by traditional structural models calibrated in that paper. In Model 3, we add the countercyclical $L$ but keep a constant risk aversion.
The implied average spread almost doubles, but the standard deviation of spreads remains very low. This result sheds light on the importance of countercyclical loss at generating high levels due to the covariance with the pricing kernel. However, it has a minor effect at producing reasonable variations of spreads. Next, we add countercyclical $\gamma$ but keep $L$ at its long run average. The implied mean and the standard deviation of Baa-Aaa spreads are 52 bp and 12.5bp, both of which are higher than in model 3, but still falls well below their empirical levels at 109 bp and 41 bp, respectively. This level is comparable to the CCDG result using habit only. Finally, in the full model (model 5) which also incorporates countercyclical loss rate, the matches for both the average and the standard deviation of credit spreads are near perfect. This result again emphasizes the importance of both countercyclical $\gamma$ and $L$.

By comparison, time variations in $\gamma$ is more important than time variations in $L$ to generate credit spread volatilities. The effect of $\gamma$ coupled with $L$ has a larger impact than that from their multiplication (the variation increases dramatically using both relative to each one of them considered separately). This is clearer if one notices how spread covaries with different level of loss rate and risk aversion as plotted in Figure 4. Baa-Aaa spread starts from a minimum of 20 bp (reflecting expected loss) and exceeds 450 bp over very severe economic conditions. Note that spread levels indeed reach above 400 bp during the recent credit crunch. However, we want to caution that during this period the differential liquidity component was probably high between Aaa and Baa.

To give the more direct interpretation about the above matches, we pursue the closed-form expression for $SP^{BA}$ available in our setup, and discuss separately each of components in the formula that affect the the first two moments of credit spreads. We leave this discussion to Appendix C.

**B.2. Credit spread term structure**

So far we’ve documented matches on 5 years Baa-Treasury and Aaa-Treasury spreads, and also on 4-year Baa-Aaa spreads. We now turn our attention to the performance of the model in matching the observed spread term structure. Again, to focus exclusively on the default compensation we restrict our analysis on Baa-Aaa spreads. We get slightly a higher level because of the different specification of $\gamma$ in our model versus theirs (MSV vs CC).
calibrate our full model to the term structure of objective default probabilities. We collect cumulative default probabilities for maturities ranging from 1 to 20 years for Baa and Aaa bonds from Moody’s handbook. In particular, we use two default rate averages: 1970–2008 and 1920–2008. The corresponding model implications for maturities up to 10 years are plotted in the top left and the top right panel of Figure 5, respectively. The bottom two panels of the same figure do this exercise for maturities up to 20 years. We split both to help the reader focus on each part of the curve at one time.

First, the period from 1920 to 2008 contains much more economic downturns than the one from 1970 to 2008. Consequently, the implied credit spreads are higher in the right panels of Figure 5 compared with spreads in the left panel for each maturity.

Second, the implied short term spread is strictly positive and sizable in our full model for the following three reasons. First, relating the default of IG bonds to the peso component in consumption makes default an inaccessible event in our setup. This is a desirable property of the incomplete information framework, which is also behind the success of nonzero short term spread in Chen and Kou (2009) and Duffie and Lando (2001). In addition, jump to default for IG bonds coincides with states of high marginal utility. Furthermore, The risk aversion also peaks around default times. All these three effects help generate not only non-zero spread for short term but also a sizable level that matches the data. Indeed, the average 1-year Baa-Aaa spread using data from Bloomberg during 1980–2008 is around 73bp. This level falls into the range of our model implied short term spread (51bp for the shorter sample and 84bp for the longer sample).

One crucial insight from the above result is that habit models augmented with peso component plus the recognition that IG bonds tend to default during bad macro conditions do match the short term spread quite well. This is in contrast to long run risk models. As explained in footnote 2, the real term structure of interest rates in these models is downward sloping which tends to generate higher spreads in the long end. The same effect influences the short end of the term structure, but inversely this time. To get a sense about the magnitude of short term credit spreads implied from long-run risk framework, we calibrate a long run risk model to Moody’s objective probabilities averages and calculate bond prices for 1 year maturity. We find that the model generates much lower Baa-Aaa spread (23 when calibrated to 1970 to 2008 and 39 when calibrated to 1920 to 2008), which fall short of the data values.
Third, traditional structural models usually generate high steepness of the credit spread term structure. As mentioned in Duffie and Lando (2001), this is due to the fact that, for models producing zero short term spread in the limit, a steep slope is required to catch up with the observed spreads for longer term. As noted above, our full model matches the short term spread quite well. Consequently, as illustrated in all four panels of Figure 5, it does not induce a severe steepness in order to get reasonable spread levels for medium maturities.

Fourth, our model generates an upward sloping term structure for shorter time horizon, which is particularly true when $\tau < 5$ years for both time periods. Interestingly, our model produces both upward and humped shaped term structure after maturities higher than 5 years, dependent upon which objective default probabilities the model has been calibrated to. This is an empirical fact also documented in Jarrow et al (1997) who find that term structures are upward sloping for credits rated A or better. They become slightly humped for Baa rated credits.

Finally, note that in the long run the term structure tends to flatten which is an observed empirical fact. This is a desirable property of modeling default as in the incomplete information setup. In contrast, modeling default as the first hitting time of an asset boundary often lead to a convergence to zero in the long end.

B.3. Time series performance

Being able to match the second moments of Baa-Aaa spreads for 4 years level hints toward the ability of our model at capturing the time series property of spreads. To investigate along this dimension, As in CCDG, we obtain the innovation of the historical log consumption growths from St Louis Fed and Campbell Shiller’s website. Following CC and MSV. We use this innovation to construct the time series of the surplus ratio and loss rate using (2.1)–(2.2) and (2.12), respectively. Given the time series of $S$ and $L$, we back out the model implied spreads and compare them to historical data. The top two panels of Figure 6 plot together the model-implied credit spreads and their changes vs. the historical values. We do similar exercise with variation in $L$ shut down, and plot in the bottom two panels of the same figure the same model implications vs. their data counterparts.

We first focus on the full model. Time series implied from the model and the data exhibit similar dynamics throughout the 92-year period. Quantitatively, the model implied spreads mimic well the historical mean (130bp vs 126bp), the standard
deviation (83.7bp vs. 80.0bp), the minimum (46.5bp vs 34bp), and the maximum (460bp vs 510bp), and the two series are correlated by 62%. In terms of changes of the credit spreads which are plotted in the top right panel, the two series are correlate by 23.3% for the full sample.

Turning to the bottom two panels, we find that the implied dynamics from the data and the sub-model without variations in $L$ are still quite similar. Indeed, the correlation for credit spreads and changes of spreads are 58% and 23.4% as compared to 62% and 23.3% with the full model. The reason is that processes of loss rate and surplus ratio are both driven by consumption shocks in our setup. Consequently, the sub model still capture most of macro level variations. However, the sub model fails to capture extremely high spreads during the Great Depression and the recent financial crisis. Indeed, the maximum credit spread is now 321bp as compared to 460bp in the full model and 510bp in the data. This exercise again demonstrates that both countercyclical $\gamma$ and countercyclical $L$ are important for our model’s pricing performance.

C. Joint pricing

In this subsection, we investigate the links between different financial markets from the lens of our model. Panel C&D of Table I report part of model implications on the pricing of aggregate equity and its derivatives. In particular, the model matches the low short risk free rate in terms of both the average level and the standard deviation, the high equity premium, the excess stock return volatility, and the 10% smirk premium, measured as the difference in B/S-vol between 10% OTM put option and ATMs. For pricing equity and options separately using habit formation plus a peso component, we refer readers to Du (2011) for the more details.

As commented in the introduction, the joint pricing of defaultable bonds and equity index options is natural and our model provides a coherent disaster story that unifies their pricing. To give the graphical illustration, we vary the absolute consumption jump size $|b|$, and plot in the top two panels of Figure 7 the implied average Baa-Aaa credit spread and average smirk premiums, measured as the price difference between 10% OTM puts and ATMs. Both variables rise monotonically with $|b|$. For defaultable bonds, a larger $|b|$ means higher price of default risk, or equivalently, higher default intensity under the $Q$ measure, which drives up the default-risk pre-
mium and hence the credit spread. On the other hand, larger $|b|$ implies higher jump risk premium for the aggregate equity, which is reflected as the large smirk premium implicit in its derivatives.

To give a more direct comparison, we plot in the bottom left panel of the same figure the average credit spread against the average smirk premium as we vary $|b|$. We know from Du (2011) that habit formation in the absence of a peso component can still produce a positive smirk premium, but the level is off by 50%. In addition to confirming this finding, we show that in the absence of the peso component, the implied credit spread is less than one-fourth of the observed level. Both rise in lockstep as we increase the consumption jump severity until they reach their respective empirical level at our base case consumption jump calibration. Thus, we argue that a small disaster component is crucial to provide a link between the credit and the option markets. The intuition is straightforward: index options constitute the prime liquid market for insurance against systematic jumps, precisely the type of jumps that holders of defaultable bonds are exposed to. For completeness, the bottom right panel of Figure 7 plots the average credit spread against the average equity premium as we vary $|b|$. By comparison, allowing for the peso component and relating defaults of IG bonds to these severe consumption shocks have greater impacts on credit spreads than on the other two assets.

The above finding justifies the empirical findings that smirk can be used to explain credit spreads. CGM and CDM, among others, use smirk to proxy for unobservable jumps in asset values and changes in smirk as severe changes in macroeconomic conditions. They find that the smirk explain significantly both the level and changes credit spreads. However, given our setup, we note that smirk cannot be viewed as an exogenous variable for spreads since both smirks and spreads are endogenously affected/determined by the exposure of their respective markets to economic disasters. Another more important point is that our positive relationship, as illustrated in Figure 7, is true only for the given state of the economy. In order to understand how this relationship is affected by economic cycle, we carry on a comparative statics with respect to the states of the economy as proxied by different levels of surplus ratio.

The top two panels of Figure 8 plots the state dependences of the four year Baa-Aaa spreads and the option smirk premiums, where the default loss is fixed at its long run average. As discussed previously, credit spread is clearly countercyclical. Indeed, recently many economists argued that credit spread index (or CDX in short)
can be used to forecast economic cycles. The smirk premium, however, is a different story: it is decreasing at very low surplus ratio $S$, and becomes increasing for higher $S$. This is because an increase of the surplus ratio has two offsetting effects on the implied smirk premium. First, it decreases the severity of consumption jumps under the pricing ($Q$) measure which gives the agent less incentives to hedge, hence the lower option premium for the given diffusive stock volatility $\sigma_{Pt}^p$. Second, it decreases $\sigma_{Pt}^p$ which increases the relative importance of jumps in the total stock price variations. As a result, the agent has stronger incentives to buy insurance from OTM puts which drives up the smirk premium for the given $\lambda^Q$ and $J^p_{Pt}$. The first effect dominates when $S$ is small, hence the higher premium as the economy gets worse. However, for relatively large $S$, the second effect dominates leading to the rising smirk premium in $S$. Thus, while it is widely acknowledged that credit spreads proxy for macroeconomic conditions, our analysis suggests that this is not the case for smirk premium. In particular, we provide a strong argument that among competing financial variables, credit spread seems to be the better proxy for the states of the economy.

Another insight from our model is that one has to be cautious when running regression of credit spreads on smirk premiums, since their relation is dependent on what economic states the sample covers. To provide an intuitive reconciliation, we decompose credit spreads into its own two big components: expected loss versus bond risk premium. It is intuitive that smirk premium should be more related to the risk premium component in credit spread rather than total spread. This fact is shown clearly in the bottom two panels of Figure 8: the premium component rises monotonically with the smirk under both the low and the high levels of $S$.

As explained previously, our model’s resolution of credit spread puzzle hinges on a peso component, which drives jumps in the pricing kernel, and relating defaults to these severe consumption shocks. The magnitude of consumption jumps used in this paper, hence the implied magnitude of pricing kernel jumps $J_A$, is estimated by Du (2011) using the S&P 500 index option data in a nearly 20-year period. In our setup, pricing kernel jump magnitude also controls the price of default risks which generates the wedge in default intensity between the $Q$ and the $P$ measure. In particular, default intensity ratio between the two measures is equal to $1 + J_A$. On average, this ratio is about 3.1 which is lower than the one reported by CCDG (5.28 for Baa and 14.8 for Aaa). However, this number is in line with the empirical findings in Berndt, Duffie,
Douglas, Ferguson and Schranz (2008) and Elkamhi and Ericsson (2009). This result shows that our model performance at generating the observed credit spreads is not due to a higher calibrated default premium. In addition, it provides further evidence about links between option and credit market.

Panel B of Table II reports the state-dependences of the default intensity ratio which is countercyclical. In particular, low surplus ratio $S$ implies high $1 + J_A$ denoting high compensation per unit of bond price jump attributed to defaults. CCDG emphasize the importance of a countercyclical default rate under $P-$measure driven by the countercyclical default boundaries which also generate in their model the large default rate wedge between the two measures. In our setup, default intensity under the $P$ measure for IG bonds is not stochastic but correlated with consumption jumps. Default risks are priced through the covariance between default intensity and the disaster intensity. We conclude that a time-varying default rate under $P$ measure is not required for a model to resolve the credit spread puzzle. What is necessary is the covariance with the pricing kernel. It is also not necessary to have a time varying $P$ intensity to generate a stochastic intensity under the $Q$ which is required to match the data. In fact, in our habit setup, $Q$ intensity of default rate $\lambda^Q$ becomes highly time varying due the volatility of the surplus ratio even though we have no variation in $\lambda^P$, and our implied ratio matches closely the data.

VI Robustness

We start this robustness section with the sensitivity of the model-implied spreads to some key parameters. Specifically, we choose to focus on $\alpha$, $\alpha_L$ and $|b|$: the first two controls the variations of $\gamma$ and $L$ in response to consumption innovations, and last one

\[ \rho (dN^D_t, dN_t) = \frac{\text{Cov}(dN^D_t, dN_t)}{\sqrt{\text{Var}(dN^D_t)} \sqrt{\text{Var}(dN_t)}} = \frac{\text{Cov}(dN^D_t, dN^D_t + dN^D_t)}{\sqrt{\text{Var}(dN^D_t)} \sqrt{\text{Var}(dN_t)}} = \frac{\text{Cov}(dN^D_t, dN^D_t)}{\sqrt{\text{Var}(dN^D_t)}} = \sqrt{\frac{\lambda^D}{\lambda^D \lambda^D + \lambda^D}} = \sqrt{\frac{\lambda^D}{\lambda^D}} \]

is the Poisson process independent of $dN^D_t$ which models the arrivals of economic disasters that do not induce defaults for the given credit class.

Berndt et al (2008) estimate the Black Karazinski model for the $P$-intensity using the expected default intensity. A careful look at their estimation for CDS firms shows that the variation part of the $P$-intensity is indeed minor compared to the $Q$-intensity of default.
denotes the absolute log consumption jump size. The reason for the choice is apparent given the above analysis: three components play important role, i) accounting for the peso problem ii) time varying risk aversion and iii) countercyclical loss rates. We vary each of the three parameters by up to 30% in both directions relative to their base case levels calibration, and plot the implied average and standard deviation of 4-year Baa-Aaa spreads in Figure 9.

The directions of changes are as expected. By comparison, $|b|$ has the largest impact which is followed by $\alpha$. This result re-emphasizes the importance of accounting for the peso problem, not only at explaining aggregate equity and options but also credit spreads for IG bonds. The importance of the jump component can also be inferred from the long run risk literature. Using a model that is essentially the discrete time version of Bansal and Yaron (2004) with jumps in the form of regime switching, Chen (2010) generates a high average credit spread for 10-year Baa bonds. Like our model, Chen emphasizes the effect of a large jump risk premium on the pricing of defaultable bonds. In particular, by shutting down jumps, Chen reports a roughly 50% reduction in the generated spread, which is very much in line with our numerical results.

We next turn our attention to the robustness concerning our specification about habit formation. To answer the question whether our results are driven by the specific MSV habit, we use instead the specification by Santos and Veronesi (2010; SV) which is close to the original CC habit with an extra parameter $\phi$ controlling the curvature of the utility$^{14}$. We solve for all asset valuations in this new framework$^{15}$. We find that SV-habit augmented with a peso component is also able to match the high credit spreads while keeping close matches for aggregate equity and its derivatives. Consequently, we conclude that our results are not sensitive to changes of habit specification.

When implementing the model, we follow Bakshi, Madan, and Zhang (2006) who

---

$^{14}$We use $\phi$ instead of $\gamma$ as in SV to avoid the abuse of notation. At the first sight, one may argue that having an extra parameter will surely enhance the model performance to capture prices in-sample. However, this is not the case in our exercise. The reason is that we fit our model to only the S&P 500 option data to infer habit and peso component parameters. The matching of credit spread is similar to an out-of-sample exercise which becomes even harder with richer parameterization. Therefore, choosing SV instead of MSV and showing that our results still hold, demonstrates that our model performance is not completely dependent upon the MSV habit that we focus on in this paper.

$^{15}$We consider various $\phi$ under SV habit. To save space we do not report the results (closed form solutions, Figures and Tables), which are available upon requests.
document empirically that high grade bonds’ defaults are mainly explained by systematic factors. As the robustness check, we want to provide insights concerning the sensitivity of our model performance to idiosyncratic shocks. Unfortunately, there are not many observed defaults for IG bonds, and even less default cases that do not coincide with recessionary times for these credit classes. We thus resort to a numerical experiment designed to answer the following question: how big the idiosyncratic fraction of default with respect to total default has to be in order to achieve a perfect match of the observed data?

We use the 5-years CDS data to perform the exercise. Given that our model generates Aaa spread that equals the average CDS spread almost exactly, the implied idiosyncratic fraction for this credit class is essentially zero. For Baa rating, which is the lowest class in the IG category, our model-implied fraction of idiosyncratic defaults is about 1 out of 5, and this result is plotted in the bottom left panel of Figure 9. Taken together, we conclude that our modeling choice of relating defaults of high rated bonds to severe economic conditions is supported by the data.

Including Aaa and Baa, we perform the above exercise for 8 rating classes (Aaa to Caa) for which Moody’s objective default probability and average CDS spreads are available. We find that the fraction of default happenning independent of bad economic conditions increases as we go down in credit quality: idiosyncratic component is indeed minor for all IG bonds, and it gets higher for speculative grade to become the most important driver of default for very low credit rating. An important insight from this exercise is that our model produces a well documented empirical fact: the implied ratio of default intensity between $P$ and $Q$ measure decreases with credit quality (See, among others, Elkamhi and Ericsson (2008), Berndt et al (2008) and Coval et al (2009)). To give the graphical impression, we plot in the bottom right panel of Figure 9 the default intensity ratio against the fraction of idiosyncratic default, which shows a downward sloping pattern as idiosyncratic component becomes larger for the lower credit classes.\footnote{To see the result mathematically, denote by $\lambda_D$ and $\lambda_D(Q)$ the total default intensity under $P$ and $Q$ measure, respectively, for the given credit class. Following the decomposition, we write $\lambda_D = \lambda^{sys}_D + \lambda^{idio}_D$. Since idiosyncratic default is not priced, $\lambda_D(Q) = (1 + J\lambda) \lambda^{sys}_D + \lambda^{idio}_D$, where the gross jump size of the pricing kernel, $1 + J\lambda$, generates the wedge in systematic defaults between $P$ and $Q$ measure. The intensity ratio is thus $\frac{(1+J\lambda)\lambda^{sys}_D + \lambda^{idio}_D}{\lambda^{sys}_D + \lambda^{idio}_D}$ which is clearly decreasing in $\lambda^{idio}_D$. In particular, it is $1 + J\lambda$ when $\lambda^{idio}_D = 0$ and 1 when $\lambda^{idio}_D = \lambda_D$.} This result sheds light on the performance of our model at reproducing prices at all spectrum of credit quality. In this study, to save
space, we focus only on high grade firms since this is the class that is traditionally
difficult to price.

VII Conclusion

This paper proposes a preference-based general equilibrium model capable of jointly
explaining the high credit spread for IG bonds and the pronounced volatility smirk
for the pricing of equity index options. We assume non-time-separable preferences
induced by habit formation, and the lognormal consumption process subject to a
small-probability jump that models economic disasters. The negative reaction of risk
aversion to changes in aggregate consumption creates a channel by which consumption
innovations elicit extra diffusions and jumps in both the pricing kernel and the equity
price. Given that deep OTM puts are more sensitive to jumps than ATMs, such
premium translates into the high smirk premium in the pricing of aggregate equity
derivatives. On the other hand, the high price of jump risks translates into price
of default risks when defaults are triggered by disasters. When combined with the
dramatic downward jumps of bond price upon default, this effect gives rise to the large
default risk premium even at the small default rates, hence the high credit spreads.

One important insight from this study is that accounting for a small probability
disaster component is crucial to provide a link between the three markets. In fact,
both IG credit spread and option smirk premium are severely underpriced when the
peso component is shut down, and they both reach their respective empirical level at
our base-case consumption jump calibration. The peso component, together with the
incomplete information approach that relates IG bond defaults to severe economic
conditions, also enhances our model performance with respect to that of CCDG in
terms of capturing the observed shapes of credit spread term structure. Furthermore,
with the implied slightly upward-sloping yield curve for default-free discount rates,
our model improves upon the pricing of defaultable securities compared to the long-
run risk models, while simultaneously capturing salient features in other markets.

While we provide results on time series of credit spread, the main focus of the
present paper is to jointly price defaultable bonds and equity index options in terms
of the average levels. An interesting extension is to study variations of credit spreads
in relation to price dynamics of other assets. Empirically, several research articles
(e.g., EDAM; CGM; Schaefer and Strebulaev, 2008) report that corporate bond prices
are related to a number of market-wide factors such as excess returns (e.g., equity premium, Fama-French HML factors) or measures of volatilities (e.g., VIX), even after controlling for all variables that standard credit risk models claim are sufficient to determine spreads. These empirical results can be explained by our model in which credit spreads, together with moments related to the aggregate equity market, are all driven by time-varying risk aversions serving as the state. While the present paper does not touch cross-sectional pricing of equity, the predicting power of FF factors also seem explainable within the habit formation framework.\textsuperscript{17} Taken together, our model (or its variations) appears to be a promising tool for studying the joint dynamics/cross section in equity and bond markets, which could deepen our understanding of market integration.

\section*{Appendix}

\textbf{A. Outline of the proof of Proposition 2}

The main theoretical result of this paper is Proposition 2 which gives, at the presence of habit formation, the closed form valuation of a zero-coupon defaultable bonds under the reasonable RFV arrangement. The proof relies on a three-factor application of the general framework of "linear factor processes augmented with structural changes" (LS) proposed by Du and Elkamhi (2011). Below, we briefly outline its proof, and readers are referred to a separate technical appendix for the more detailed procedures.

\textbf{A.1. LS and mathematical notations}

LS as proposed in Du and Elkamhi (2011) provides analytical treatment for stocks and bonds under a general class of asset pricing models with regime-dependent affine factor processes. Switches among regimes, which are referred to as structural changes in LS, have very flexible economic interpretations such as business cycle variations, abrupt policy changes, migrations of credit ratings including the claim of bankruptcy, etc. An important advantage of LS is that it permits closed form valuations for both default-free and defaultable bonds with an arbitrary number of factors and regimes. We leave the detailed description of LS is to a technical appendix. Du and Elkamhi (2011) consider two types of LS, LS-I and LS-II, dependent upon the

\textsuperscript{17}For example, Santos and Veronesi (2010) show that under a large cross-sectional dispersion in cash-flow risks across firms, habit formation model can generate not only a sizable value premium but also a countercyclical value premium. In other words, time-varying risk aversion due to habit formation can drive the observed dynamics of cross-sectional stock returns, hence the exposure of credit spread to value premium.
different specifications about payoff function and factor processes. For our paper, LS-II is more relevant. In fact, the proof of Proposition 2 lies in an application of the three-factor two-regime case of LS-II, where the three factors are a function of consumption, stochastic risk aversion, and stochastic loss rate; the two regimes are the non-default and the default regime.

Before proceeding to the proof, it is helpful to introduce some mathematical notations. Let $1_{2 \times 1}$, $I_{2 \times 2}$, $0_{2 \times 1}$, and $0_{2 \times 2}$ denote the 2 by 1 unit vector, the 2 by 2 identify matrix, the 2 by 1 zero vector, and the 2 by 2 zero matrix, respectively. Let $\odot$ denote the element by element multiplications. For $z \in \mathbb{R}^2$, $\text{diag}(z)$ denotes an 2 by 2 diagonal matrix with its $ii$th entry equaling the $i$th entry in $z \in \mathbb{R}^n (i = 1, 2)$. For $\forall x, y \in \mathbb{R}^2$, we have the identity

$$x \odot y = \text{diag}(x)y \text{ for } \forall x, y \in \mathbb{R}^2. \quad (A.1)$$

For $X \in \mathbb{R}^{2 \times 2}$, $e^X$ denotes the matrix exponential defined by:

$$e^X \equiv \sum_{k=0}^{\infty} \frac{1}{k!} X^k. \quad (A.2)$$

An important notation in LS-II that will be used in the following is the so called "$\{0,1\}$-multi-index". To understand it, we first introduce the multi-index notation. A $K$-dimensional multi-index is a $K$-tuple, $\tilde{\alpha} \equiv [\tilde{\alpha}_1, ..., \tilde{\alpha}_K]'$, of non-negative integers with its norm defined by $|\tilde{\alpha}| \equiv \tilde{\alpha}_1 + ... + \tilde{\alpha}_K$. For a vector $x \equiv [x_1, ..., x_K]' \in \mathbb{R}^K$, the power $x^{\tilde{\alpha}}$ denotes $\prod_{k=1}^{K} x_k^{\tilde{\alpha}_k}$. A $\{0,1\}$-multi-index $\alpha$ is a special version of the multi-index in that its entries are either 0 or 1. As a result, $0 \leq |\alpha| \leq K$. In LS-II (assuming a single regime), the payoff function is usually written as:

$$\Phi_t(x_t) = \sum_{|\alpha|=0}^{K} \theta_{\alpha} x_t^{\alpha}, \quad (A.3)$$

where the sum is across all possible products of $x$ without repetition; $\theta_{\alpha}$ is the corresponding coefficient. For example, when $K = 3$, $\Phi_t(.)$ in (A.3) degenerates

$$\Phi_t(x_t; s_t) = \theta_{[0,0,0]'}(s_t) x_{1t} + \theta_{[1,0,0]'}(s_t) x_{2t} + \theta_{[0,1,0]'}(s_t) x_{3t} + \theta_{[1,1,0]'}(s_t) x_{1t}x_{2t}$$

$$+ \theta_{[1,0,1]'}(s_t) x_{1t}x_{3t} + \theta_{[0,1,1]'}(s_t) x_{2t}x_{3t} + \theta_{[1,1,1]'}(s_t) x_{1t}x_{2t}x_{3t}. \quad (A.3)$$

In the following derivations, we consider a generalized version of the loss rate process (2.13):

$$dL_t = k_L(\bar{L} - L_t)dt - \alpha_L \sigma dB_t - \alpha_L b(dN_t - \lambda dt), \quad (A.4)$$
where $\tilde{B}_t$ is another standard Brownian which correlated with the Brownian driving diffusive consumption shock, $B_t$, by $\hat{\rho}$. This specification allows for idiosyncratic shock in addition to consumption shock to drive $L_t$. Note we maintain the assumption that the jump of $L_t$ is completely driven by severe economic state captured by disasters.

### A.2. Closed form valuation under TP

For the proof of Proposition 2, we start with the derivation of closed form valuation for defaultable bonds under the terminal payment (TP) arrangement which is described in Section II.C. There are three reasons for doing this. First and foremost, it provides the basis to derive the closed form under the more reasonable RFV arrangement. Second, we need results under TP to provide intuitions about discount rate effect on credit spreads which is discussed in details in Appendix C. Third, it is useful to demonstrate the intuitions for the closed form formulas.

Let $P_{b,TP}^{b,TP} \in \mathbb{R}^2$ be the prices of a zero-coupon defaultable bonds under TP with $\tau$ period to expiration, where $P_{b,TP}^{b,TP}$ (1) and $P_{b,TP}^{b,TP}$ (2) denote prices conditional on no-default (the first) and the default (the second) regime. For $i \in \{1, 2\}$, normalizing bond face value, and by the definition of TP,

$$P_{b,TP}^{b,TP} (i) = \begin{cases} E_t \left[ \frac{\Lambda_T}{\Lambda_t} | s_t = i \right] & \text{if } s_T = 1 \\ E_t \left[ \frac{\Lambda_T}{\Lambda_t} (1 - L_T) | s_t = i \right] & \text{if } s_T = 2 \end{cases},$$

(A.5)

where $\Lambda_T$ and $L_T$ denote the pricing kernel and the loss rate at the expiration date $T$; we’ve used that default regime is absorbing. Using the expression for pricing kernel derived in Section II.A, we obtain from the above equation:

$$P_{b,TP}^{b,TP} (i) = \frac{C_t}{\gamma_t} \left( E \left[ e^{-\rho(T-t)} \theta^c (s_T) C_T^{-1} \gamma_T | s_t = i \right] + E \left[ e^{-\rho(T-t)} \theta^s (s_T) C_T^{-1} \gamma_T L_T | s_t = i \right] \right),$$

(A.6)

where

$$\theta^c = \begin{bmatrix} \theta^c (1) \\ \theta^c (2) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \theta^s = \begin{bmatrix} \theta^s (1) \\ \theta^s (2) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$  

(A.7)

The following proposition summarizes the closed form for $P_{b,TP}^{b,TP}$.

**Proposition 2’**: Under the TP arrangement,

$$P_{b,TP}^{b,TP} = \alpha_1 b,TP (\tau) + \alpha_2 b,TP \left( \frac{1}{\gamma_t} \right) + \alpha_3 b,TP (\tau) L_t + \alpha_4 b,TP (\tau) \frac{L_t}{\gamma_t},$$

(A.8)

where

$$\alpha_1 b,TP (\tau) = e^{-\tau A_{b,1}^{b,c} \theta^c} + \int_0^\tau e^{-(\tau-u)} A_{1,0}^{b,s} B_{b,1}^{b,s} (u) du.$$

(A.9)
\[ \alpha_{2,TP}^b(\tau) = \int_0^\tau e^{-(\tau-u)A_{[1,0]'}}B_{[1,0]'}(u)du + \int_0^\tau e^{-(\tau-u)A_{[1,0]}'a}B_{[1,0]}'(u)du \quad (A.10) \]

\[ \alpha_{3,TP}^b(\tau) = e^{-\tau A_{[1,1]}'b} \theta^s \quad (A.11) \]

\[ \alpha_{4,TP}^b(\tau) = \int_0^\tau e^{-(\tau-u)A_{[1,0]'}}B_{[1,0],1}'(u)du \quad (A.12) \]

In (A.9)–(A.12); \( \theta^c \) and \( \theta^s \) are defined by (A.7);

\[ A_{[1,1]}'^{b,c} = a_{11}^{b,c}I_{2\times 2} - e^{-b} (1 - \alpha b) \Gamma^{sys} - \Gamma^{idio} \quad (A.13) \]

\[ A_{[1,0]}'^{b,c} = a_{10}^{b,c}I_{2\times 2} - e^{-b} \Gamma^{sys} - \Gamma^{idio}, \quad (A.14) \]

\[ B_{[1,0]'\tau}^{b,c} = b_{10}^{b,c}e^{-\tau A_{[1,1]}'b} \text{idio} + \alpha \beta be^{-b} \Gamma^{sys} e^{-\tau A_{[1,1]}'b} \theta^c, \quad (A.15) \]

\[ A_{[1,1]}'^{b,s} = a_{111}^{b,s}I_{n\times n} - e^{-b} (1 - \alpha b) \Gamma^{sys} - \Gamma^{idio}, \quad (A.16) \]

\[ A_{[1,0]}'^{b,s} = a_{101}^{b,s}I_{n\times n} - e^{-b} \Gamma^{sys} - \Gamma^{idio}, \quad (A.17) \]

\[ A_{[1,1]}'^{b,s} = a_{110}^{b,s}I_{n\times n} - e^{-b} (1 - \alpha b) \Gamma^{sys} - \Gamma^{idio}, \quad (A.18) \]

\[ A_{[1,0]}'^{b,s} = a_{100}^{b,s}I_{n\times n} - e^{-b} \Gamma^{sys} - \Gamma^{idio}, \quad (A.19) \]

\[ B_{[1,0]'\tau}^{b,s} = b_{10}^{b,s}e^{-\tau A_{[1,1]}'b} \theta^s + \alpha \beta be^{-b} \Gamma^{sys} e^{-\tau A_{[1,1]}'b} \theta^s, \quad (A.20) \]

\[ B_{[1,1,0]'\tau}^{b,s} = b_{110}^{b,s}e^{-\tau A_{[1,1]'b} \theta^s - e^{-b} (1 - \alpha b) \alpha \beta b \Gamma^{sys} e^{-\tau A_{[1,1]}'b} \theta^s, \quad (A.21) \]

\[ B_{[1,0]'\tau}^{b,s} = b_{100}^{b,s,1} \left( \int_0^\tau e^{-(\tau-u)A_{[1,0]'a}b}B_{[1,0,0]'\tau}(u)du \right) + b_{100}^{b,s,2} \left( \int_0^\tau e^{-(\tau-u)A_{[1,0]}'a}B_{[1,0,0]'\tau}(u)du \right) + b_{100}^{b,s,3} e^{-\tau A_{[1,1]}'b} \theta^s + e^{-\gamma_b \alpha \beta b} \Gamma^{sys} \left( \int_0^\tau e^{-(\tau-u)A_{[1,0]'a}b}B_{[1,0,0]'\tau}(u)du \right) - e^{-\gamma_b \alpha \beta b} \Gamma^{sys} \left( \int_0^\tau e^{-(\tau-u)A_{[1,0]'a}b}B_{[1,0,0]'\tau}(u)du \right) \quad (A.22) \]

where \( \Gamma^{sys} \) and \( \Gamma^{idio} \) model systematic and idiosyncratic defaults, respectively, which are described in Section II.B. In (A.13)–(A.15), \( a \) and \( b \) by (A.38)–(A.40). In (A.16)–(A.22), \( a \) and \( b \) are given by (A.28)–(A.36).

**Proof of Proposition 2**: We map the above problem into LS-II by rewriting (A.6) as

\[ P_{L,T}^{b,TP}(i) = \frac{C_i}{\gamma_i} \left[ \psi_b^c(x_t; i) + \psi_b^c(x_t; \bar{i}) \right], \quad (A.23) \]

where

\[ \psi_b^c(x_t; i) \equiv \left[ e^{-\rho(T-t)}x_{1T}x_{2T}\theta^c(s_T)du|s_t = i \right] \quad (A.24) \]

41
\[ \psi_b^s(t; \bar{a}; \bar{b}) \equiv E \left[ e^{-\rho(T-t)}x_{1T}x_{2T}x_{3T}\theta^s(s_T)du|\sigma_t = \bar{t} \right]. \] \hspace{1cm} (A.25)

In (A.24)–(A.25),
\[ x_{1t} \equiv C_{\bar{t}}^{-1}; \ x_{2t} \equiv \gamma_t; \ x_{3t} \equiv L_t \] \hspace{1cm} (A.26)

which serve as the three factors related to pricing. By Ito’s lemma with jumps,
\[ dx_{1t}/x_{1t} = \left( -\mu + \frac{1}{2}\sigma^2 \right) dt - \sigma dB_t + (e^{-\gamma b} - 1) dN_t, \]
\[ dx_{2t} = [k(\bar{\gamma} - x_{2t}) + \alpha(x_{2t} - \beta) b\lambda] dt - \alpha (x_{2t} - \beta) \sigma dB_t - \alpha (x_{2t} - \beta) bdN_t, \]
\[ dx_{3t} = [k_L(\bar{L} - x_{3t}) + \alpha_L b\lambda] dt - \alpha_L \sigma d\bar{B}_t - \alpha_L bdN_t. \]

We start with the valuation of \( \psi_b^s(.). \) By applying the three-factor case of LS-II (with details in the technical appendix), we obtain
\[ \psi_b^s(x_t) \equiv [\psi_b^s(x_t; 1), \psi_b^s(x_t; 2)]' = \sum_{\alpha \not\equiv [1,1,1]'} \eta_\alpha(\tau) x_t^\alpha + \eta_{[1,1,1]'}(\tau) x_{1t}x_{2t}x_{3t} \]
\[ = \sum_{\alpha \not\equiv [1,1,1]'} \left( \int_{0}^{\tau} e^{-(\tau-u)A_b^{h_s}} B^{h_s}_\alpha(u)du \right) x_t^\alpha + e^{-\tau A_{h_s}^{[1,1,1]'}} \theta^s x_{1t}x_{2t}x_{3t}, \] \hspace{1cm} (A.27)

where \( \alpha \) denotes the \( \{0,1\} \)-multi-index described above. The technical appendix shows that several \( \eta_\alpha(\tau) \)s in (A.27) due to our model specification. The non-zero \( \eta_\alpha(\tau) \)s are determined by \( A_b^{h_s} \) for \( \alpha \in \{[1,1,1]', [1,0,1]', [1,1,0]', [1,0,0]'\} \) and \( B^{h_s}_\alpha \) for \( \alpha \in \{[1,0,1]', [1,1,0]', [1,0,0]'\} \), whose expressions are in (A.16)–(A.19) and (A.20)–(A.22), respectively. In addition, the technical appendix shows that \( a \) and \( b \) in the expressions of \( A_b^{h_s} \) and \( B^{h_s}_\alpha \) are given by:
\[ a_{111}^{h_s} = \rho + \mu - \sigma^2 + k - ab \lambda + k_L - \alpha \sigma^2 - \lambda (e^{-b}(1 - ab) - 1); \] \hspace{1cm} (A.28)
\[ a_{101}^{h_s} = \rho + \mu - \sigma^2 + k - ab \lambda + k_L - \alpha \sigma^2 - \lambda (e^{-b} - 1); \] \hspace{1cm} (A.29)
\[ a_{110}^{h_s} = \rho + \mu - \sigma^2 + k - ab \lambda - \alpha \sigma^2 - \lambda (e^{-b}(1 - ab) - 1); \] \hspace{1cm} (A.30)
\[ a_{100}^{h_s} = \rho + \mu - \sigma^2 - \lambda (e^{-b} - 1). \] \hspace{1cm} (A.31)
\[ b_{101}^{h_s} = k\bar{\gamma} - \alpha \beta b \lambda - \alpha \beta \sigma^2 + \lambda e^{-b} \alpha \beta b; \] \hspace{1cm} (A.32)
\[ b_{110}^{h_s} = k_L\bar{L} + \alpha_L b \lambda + \bar{\rho} \alpha_L \sigma^2 + \bar{\rho} \alpha_L \sigma^2 - \lambda e^{-b}(1 - ab) \alpha_L b. \] \hspace{1cm} (A.33)
\[ b_{100}^{h_s,1} = k\bar{\gamma} - \alpha \beta b \lambda - \alpha \beta \sigma^2 + \lambda e^{-b} \alpha \beta b. \] \hspace{1cm} (A.34)
\[ b_{100}^{h_s,2} = k_L\bar{L} + \alpha_L b \lambda + \bar{\rho} \alpha_L \sigma^2 - \lambda e^{-b} \alpha_L b. \] \hspace{1cm} (A.35)
\[ b_{100}^{h_s,3} = -\bar{\rho} \alpha_L \beta \sigma^2 - \lambda e^{-b} \alpha_L \beta b^2. \] \hspace{1cm} (A.36)

\( \psi_b^s(.) \) can be calculated similarly by applying the two-factor case of LS. In partic-
ular,
\[
\psi_b^c(x_t) = [\psi_b^c(x_t; 1), \psi_b^c(x_t; 2)]' = \sum_{\alpha \neq [1, 1]'} \eta_\alpha(\tau) x_t^\alpha + \eta_{[1, 1]'}(\tau) x_{1t} x_{2t}
\]
\[
= \sum_{\alpha \neq [1, 1]'} \left( \int_0^\tau e^{-\tau-u} A^{b,c}_\alpha B^{b,c}_\alpha(u) du \right) x_t^\alpha + e^{-\tau A^{b,c}_{[1, 1]'}\theta^c} x_{1t} x_{2t},
\]
(A.37)

where the non-zero \( \eta_\alpha(\tau) \)s are determined by \( A^{b,c}_\alpha \) for \( \alpha \in \{[1, 1]', [1, 0]'\} \) and \( B^{b,s}_\alpha \) for \( \alpha \in \{[1, 0]'\} \), whose expressions are in (A.13)–(A.14) and (A.15), respectively. The associated \( a \) and \( b \) are:

\[
a^{b,c}_{11} = \rho + \mu - \sigma^2 + k - ab\lambda - \alpha \sigma^2 - \lambda \left[ e^{-b} (1 - ab) - 1 \right],
\]
(A.38)

\[
a^{b,c}_{10} = \rho + \mu - \sigma^2 - \lambda \left( e^{-b} - 1 \right),
\]
(A.39)

\[
b^{b,c}_{10} = k\bar{\gamma} - \alpha\beta b\lambda - \alpha \beta \sigma^2 + \lambda e^{-b}\alpha\beta b.
\]
(A.40)

By combining (A.23), (A.27), and (A.37) and using the definitions of \((x_{1t}, x_{2t}, x_{3t})\) in (A.26),

\[
P_{t,\tau}^{b,TP} \equiv \left[ P_{t,\tau}^{b,TP,s}(1), P_{t,\tau}^{b,TP,s}(2) \right]' = \frac{1}{x_{1t}x_{2t}} [\psi_b^c(x_t) + \psi_b^c(x_t)]
\]
\[
= \frac{1}{x_{1t}x_{2t}} \left[ + \left( \int_0^\tau e^{-\tau-u} A^{b,s}_{[1,1,0]'} B^{b,s}_{[1,1,0]'}(u) du \right) x_{1t} + e^{-\tau A^{b,s}_{[1,1,1]'}\theta^s} x_{1t} x_{2t}
\]
\[
+ \left( \int_0^\tau e^{-\tau-u} A^{b,s}_{[1,0,1]'} B^{b,s}_{[1,0,1]'}(u) du \right) x_{1t} x_{3t} + e^{-\tau A^{b,s}_{[1,1,1]'}\theta^s} x_{1t} x_{2t} x_{3t} \right].
\]

Arrangements yield (A.8)–(A.12).

The intuition of the above derivation is simple. Since defaults induce structural changes in bond payoffs, we need to use vector integrations taking into account the interaction between non-default and default regime to obtain bond prices in both regimes. In numerical calculation, we focus on \( P_{t,\tau}^{b,TP} \equiv P_{t,\tau}^{b,TP}(1) \) denoting defaultable bond price conditional on no default under the TP arrangement.

**A.3. Closed form valuation under RFV**

We next investigate closed form valuation under RFV which is given by Proposition 2 in Section III.B. In the following, we first present the "full version" of Proposition 2 that gives expressions for the pricing coefficients. We then provides the proof based on Proposition 2 and LS.

**Proposition 2:** Denote by \( P_{t,\tau}^{b,RFV} \) the defaultable bond prices under RFV arrange-
moment. Then,

\[ P_{t, \tau}^{b, RFV} = \alpha_1^{b, RFV}(\tau) + \alpha_2^{b, RFV}(\tau) \frac{1}{\gamma_t} + \alpha_3^{b, RFV}(\tau) L_t + \alpha_4^{b, RFV}(\tau) \frac{L_t}{\gamma_t}, \quad (A.41) \]

where

\[ \alpha_1^{b, RFV}(\tau) = e^{-\tau \hat{A}_0^{b, c} \theta^c} + \int_0^\tau e^{-(\tau-u)\hat{A}_0^{b, c}} \tilde{B}_{[1,0], 0}^{b, c}(u)du \quad (A.42) \]

\[ \alpha_2^{b, RFV}(\tau) = \int_0^\tau e^{-(\tau-u)\hat{A}_0^{b, c}} \tilde{B}_{[0,0], 0}^{b, c}(u)du + \int_0^\tau e^{-(\tau-u)\hat{A}_0^{b, s}} \tilde{B}_{[0,0], 0}^{b, s}(u)du \quad (A.43) \]

\[ \alpha_3^{b, RFV}(\tau) = e^{-\tau \hat{A}_0^{b, s} \theta^s} \quad (A.44) \]

\[ \alpha_4^{b, RFV}(\tau) = \int_0^\tau e^{-(\tau-u)\hat{A}_0^{b, s}} \tilde{B}_{[1,0], 0}^{b, s}(u)du \quad (A.45) \]

In the above formula,

\[ \hat{A}_0^{b} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \hat{A}_0^{b}, \quad (A.46) \]

for \( \alpha \in \{ [1,1]', [1,0]', [1,1,0]', [1,0,0]', [1,0,1]', [1,1,1]' \} \), where \( \hat{A}_0^{b} \) are the same as those in Proposition 2';\(^{18}\)

\[ \tilde{B}_0^{b} (\tau) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tilde{B}_0^{b} (\tau), \quad (A.47) \]

for \( \alpha \in \{ [1,0]', [1,1,0]', [1,0,0]', [1,0,1]' \} \). In (A.47),

\[ \tilde{B}_{[1,0], 0}^{b, c}(\tau) = b_{10}^{b, c} e^{-\tau \hat{A}_0^{b, c} \theta^c} + \alpha \beta e^{-b \Gamma^{sys} \theta^c}, \quad (A.48) \]

\[ \tilde{B}_{[0,0], 0}^{b, s}(\tau) = b_{10}^{b, s} e^{-\tau \hat{A}_0^{b, s} \theta^s} + \alpha \beta e^{-b \Gamma^{sys} \theta^s}, \quad (A.49) \]

\[ \tilde{B}_{[1,1], 0}^{b, s}(\tau) = b_{110}^{b, s} e^{-\tau \hat{A}_0^{b, s} \theta^s} - e^{-b} (1 - \alpha b) \alpha_L b \Gamma^{sys} e^{-\tau \hat{A}_0^{b, s} \theta^s}, \quad (A.50) \]

\[ \tilde{B}_{[1,0,0], 0}^{b, s}(\tau) = b_{100}^{b, s, 1} \left( \int_0^\tau e^{-(\tau-u)\hat{A}_0^{b, s}} \tilde{B}_{[1,0,0], 0}^{b, s}(u)du \right) + b_{100}^{b, s, 2} \left( \int_0^\tau e^{-(\tau-u)\hat{A}_0^{b, s}} \tilde{B}_{[1,0,0], 0}^{b, s}(u)du \right) 
+ b_{100}^{b, s, 3} e^{-\tau \hat{A}_0^{b, s} \theta^s} + e^{-b} \alpha \beta b \Gamma^{sys} \left( \int_0^\tau e^{-(\tau-u)\hat{A}_0^{b, s}} \tilde{B}_{[1,0,0], 0}^{b, s}(u)du \right) \quad (A.51) \]

\[ - e^{-b} \alpha_L b \Gamma^{sys} \left( \int_0^\tau e^{-(\tau-u)\hat{A}_0^{b, s}} \tilde{B}_{[1,1,0], 0}^{b, s}(u)du \right) - e^{-b} \alpha \alpha_L b^2 \Gamma^{sys} e^{-\tau \hat{A}_0^{b, s} \theta^s}, \]

where the associated \( b \) are the same as those in Proposition 2'.

\(^{18}\)Note in (A.46), we do not differentiate between \( \hat{A}_0^{b, s} \) and \( \hat{A}_0^{b, c} \). The same thing for (A.47).
Proof of Proposition 2: Note that under RFV, bondholders receive the residual payment immediately upon default so that \( P_{b;RFV}^t (2) = 1 - L_t \), where \( \tau (< T) \) denotes the time of default. Intuitively, for an asset that pays \( 1 - L_t \) right now and zero thereafter, its current price is trivially \( 1 - L_t \). Translated into \( \psi^b_1 (.) \) and \( \psi^b_2 (.) \) which are defined similarly as that in (A.23), we have

\[
\psi^b_1 (x_t; 2) = -C_t^{-1} \gamma_t L_t = -x_{1t} x_{2t} x_{3t}, \quad (A.52)
\]

\[
\psi^b_2 (x_t; 2) = C_t^{-1} \gamma_t = x_{1t} x_{2t}, \quad (A.53)
\]

(A.52)–(A.53) give defaultable bond prices under RFV arrangement conditional on the default regime which are already known. In a separate technical appendix, we show that, by exploiting the solution structure of LS-II, bond prices under RFV take similar formulas as those under TP after appropriate revisions of \( A \) and \( B (\tau) \) which are described by (A.46)–(A.51).

In both Proposition 2’ and Proposition 2, the vector integration can be easily computed by methods of Gaussian quadrature (e.g., Miranda and Fackler, 2001), given that integrands are in closed forms. The accuracy of both closed forms under TP and RFV are verified via simulation exercises for various \( \tau \). With only two regimes, we also derive in a separate technical appendix more explicit expressions for \( P_{t,\tau}^{b,TP} \) and \( P_{t,\tau}^{b,RFV} \) by analytically evaluating the vector integrations. However, the obtained formulas, particularly those under RFV, are not concise and the link of pricing between TP and RFV arrangements become unclear. We thus leave the forms with the integrals in the paper and refer the reader to the technical appendix for implementation of complete closed form.

A.3.1. Proof of Proposition 1  Proposition 1 gives the price of a zero-coupon default-free bond which can be derived as a special case of Proposition with \( L_t = 0 \) and no default regime. In this case, first notice that \( \alpha_3^{b,RFV} (\tau) \) and \( \alpha_4^{b,RFV} (\tau) \) in (A.41) are both zeros. Second, \( \theta^c \) and \( \theta^s \) defined in (A.7) degenerate to 1 and 0, respectively. The latter implies \( \tilde{B}^{b,s}_{[1,1,0]} (\cdot) \) and \( \tilde{B}^{b,s}_{[1,0,0]} (\cdot) \) are both zeros from (A.47) and (A.49)–(A.51). Third, with only one regime, \( \Gamma^{gs} = \Gamma^{idio} = 0 \). Hence,

\[
\tilde{A}^{b,c}_{[1,1,0]} = \tilde{A}^{b,c}_{[1,1,1]} = a_{11}^{b,c},
\]

\[
\tilde{A}^{b,c}_{[1,0,0]} = \tilde{A}^{b,c}_{[1,0,1]} = a_{10}^{b,c},
\]

\[
\tilde{B}^{b,c}_{[1,1,0]} (\tau) = \tilde{B}^{b,c}_{[1,0,0]} (\tau) = B^{b,c}_{[1,1,0]} (\tau) = b_{10}^{b,c} e^{-\tau A^{b,c}_{[1,1,0]}},
\]

where we’ve used that \( \theta^c \) degenerates to 1. Substituting the above results into (A.41), we can derive the pricing coefficients in Proposition 1 as:

\[
\alpha_1^0 (\tau) = e^{-\tau a_{11}^{b,c}} \quad (A.54)
\]
\[ \alpha_2^0(\tau) = \int_0^\tau e^{-(\tau-u)\alpha_{10}^{bc}} b_{10}^{bc} e^{-\alpha_{11}^{bc} t} du = b_{10}^{bc} \frac{1}{\alpha_{10}^{bc} - \alpha_{11}^{bc}} \left( e^{-\tau \alpha_{11}^{bc}} - e^{-\tau \alpha_{10}^{bc}} \right) \quad (A.55) \]

**A.3.2. Independent default case** An interesting case that is discussed in Section IV.B is when defaults are independent of consumption jumps. Bond valuations under this assumption can be derived as a special case of Proposition 2 with \( \Gamma = \Gamma_{\text{sys}} \) set to zero (hence, \( \Gamma = \Gamma_{\text{idio}} \)). Conditional on no default, denote by \( P_{t,\tau}^{\text{ind}} \) the price of RFV bonds under independent default whose closed form is:

\[ P_{t,\tau}^{\text{ind}} = \alpha_1^{\text{ind}}(\tau) + \alpha_2^{\text{ind}}(\tau) \frac{1}{\gamma_t^{\text{ind}}} + \alpha_3^{\text{ind}}(\tau) L_t + \alpha_4^{\text{ind}}(\tau) \frac{L_t}{\gamma_t^{\text{ind}}}, \quad (A.56) \]

The following corollary, which is a special case of Proposition 3, summarizes the dynamics of \( P_{t,\tau}^{\text{ind}} \).

**Corollary 1:** Under independent default, the instantaneous return of RFV defaultable bond follows:

\[ \frac{dP_{t,\tau}^{\text{ind}}}{P_{t,\tau}^{\text{ind}}} = \mu_{P_t}^{\text{ind}} dt + \sigma_{P_t}^{\text{ind}} dB_t + \sigma_{P_t}^{\text{ind}} dN_t + \left[ 1 - \frac{L_t}{P_{t,\tau}^{\text{ind}}} - 1 \right] dN_{Dt}, \quad (A.57) \]

where the two processes, \( dN_{Dt} \) and \( dN_t \), are mutually independent, which model, respectively, the arrival of default and the arrival of economic disaster:

\[ \sigma_{P_t}^{b,\text{ind}} = -\alpha_2^{\text{ind}}(\tau) \frac{1}{\gamma_t^{\text{ind}}} \sigma_{P_t}^{\text{ind}} \sigma_{P_t}^{\text{ind}} - \alpha_3^{\text{ind}}(\tau) \frac{1}{\gamma_t^{\text{ind}}} \sigma_{P_t}^{\text{ind}} \left( L_t \sigma_{\gamma_t^{\text{ind}}} + \alpha_L^{\text{ind}} \right) \]

\[ \{ \gamma_t^{\text{ind}} P_{t,\tau}^{\text{ind}} - 1 = \alpha_2^{\text{ind}}(\tau) \frac{1}{\gamma_t^{\text{ind}}} \left( \frac{1}{1 + J_t} - 1 \right) - \frac{1}{\gamma_t^{\text{ind}}} \alpha_3^{\text{ind}}(\tau) \alpha_L b \} \quad (A.58) \]

where expressions for \( \mu_{t}, \sigma_{t}, \) and \( J_{t} \) are in (2.5).

From (A.57), bond risk premium under independent default case is given by

\[ BP_t^{\text{ind}} = -\sigma_{t,\tau}^{b,\text{ind}} P_t^{\text{ind}} - \lambda J_{t,\tau}^{b,\text{ind}}, \quad (A.59) \]

which will be used in discussions of Section IV.B.

**B. Negative credit spread and interest rate term structure**

In Appendix B, we analyze why our sub-models generate negative credit spreads at low loss rate \( L \) when we shut down variation in \( L \), i.e., model 1, 2, and 4 as in the five models described in Section V.A. This result is equivalent to \( V_t^0 - V_t^d < 0 \), where \( V_t^0 \)
and $V^d_t$ denote valuations of default-free and defaultable bonds. By their definitions,

$$V^0_t = E_t \left[ \frac{\Lambda_T}{\Lambda_t} \right],$$

$$V^d_t = E_t \left[ \frac{\Lambda^{\ast}}{\Lambda_t} (1 - L) 1_{(\tau^* \leq T)} \right] + E_t \left( \frac{\Lambda_T}{\Lambda_t} 1_{(\tau^* > T)} \right),$$

where $\Lambda$ is the pricing kernel; $T$ is the expiration date; $\tau^* < T$ denotes the default time. Hence,

$$V^0_t - V^d_t = E_t \left[ \left( \frac{\Lambda_T}{\Lambda_t} - \frac{\Lambda^{\ast}}{\Lambda_t} (1 - L) \right) 1_{(\tau^* \leq T)} \right]. \quad (A.60)$$

To see the possibility that $V^0_t - V^d_t < 0$, we need to digress by talking about the term structure effect. Recall the formula for real rate in the aggregate model of MSV:

$$r_t = \rho + \mu - \sigma^2 - k \frac{\gamma_t}{\gamma_t} - \alpha \frac{\gamma_t - \beta}{\gamma_t} \sigma^2.$$

There are two offsetting effects on the state-dependences of $r_t$. The intertemporal substitution effect, controlled by $-k \frac{\gamma_t}{\gamma_t}$, induces the investor to borrow during the bad state with high $\gamma_t$, which drives a high interest rate. The precautionary saving effect, controlled by $-\alpha \frac{\gamma_t - \beta}{\gamma_t} \sigma^2$, induces the investor to save more during the bad state, which drives a low rate. Adding jumps in consumption in our model, further strengthens the precautionary saving effect. In general, the substitution effect dominates even at the presence of jumps, hence the countercyclical rate. The negative correlation between default-free rate and the surplus ratio leads to positive risk premia on real bonds, and an upward-sloping yield curve.

Returning to our original discussion on the source of negative spreads under habit in the absence of countercyclical $L_t$, the upward-sloping (real) term structure for default-free bonds implies

$$-\frac{1}{T - t} \log \left[ E_t \left( \frac{\Lambda_T}{\Lambda_t} \right) \right] > -\frac{1}{\tau^* - t} \log \left[ E_t \left( \frac{\Lambda^{\ast}}{\Lambda_t} \right) \right].$$

If the slope of the term structure is sufficiently high, we may have

$$\log \left[ E_t \left( \frac{\Lambda_T}{\Lambda_t} \right) \right] < \log \left[ E_t \left( \frac{\Lambda^{\ast}}{\Lambda_t} \right) \right].$$

If in addition $L$ is small, we may further have

$$E_t \left[ \left( \frac{\Lambda_T}{\Lambda_t} - \frac{\Lambda^{\ast}}{\Lambda_t} (1 - L) \right) 1_{(\tau^* \leq T)} \right] < 0.$$
which implies, from (A.60), a negative $V_t^0 - V_t^d$. This explains (or provide sufficient conditions to) the negative spreads for the sub-models plotted on the right panels of Figure 3 when $L$ is below 0.2.

The above discussions hinge on a constant $L$. When $L$ becomes countercyclical, credit spreads are always positive. This is because even though $L_t$ starts low, it tends to be a high number upon default which tends to happen during bad times. From a different perspective, even though earlier realization of cash flow is more valuable in our model, the realized amount of cash flow tends to be low due to the countercyclical default loss. The latter effect dominates under our base case calibration, hence the sizable credit spread (around 140 bp for 5-year Baa-Treasury) in the full model even at $L_t$ set to zero when the bond contract is initiated.

To demonstrate clearly that the negative spread is exclusively due to the upward sloping term structure under habit formation, we also plot the results for Baa-Treasury spreads decompositions under the TP arrangement where upon default, the asset makes the residual payment only at the contract expiration date. Under TP, the effect of the discount factor on early realization of cash flow in the event of default in annihilated. Figure A.1 plots implication from the same five models but under the TP arrangement. Note that spreads never turn negative. It is also intuitive that for the full model, the implied spreads are lower under the TP than under RFV, which is due to the discount rate effect: conditional on default, the present value of the early realization of recovery under RFV is lower than the present value of the recovery amount at expiration.

C. Understanding Baa-Aaa spread components

In Appendix C, we study in details the closed form valuation of RFV bonds in terms of its impacts on the implied credit spread. Conditional on no defaults, we apply Eq. (3.6) to obtain for Aaa and Baa bonds:

\[
\begin{align*}
P_{t,\tau}^{Aaa} &= \alpha_1^{Aaa}(\tau) + \alpha_2^{Aaa}(\tau) \frac{1}{G_t \gamma_t} + \alpha_3^{Aaa}(\tau) L_t + \alpha_4^{Aaa}(\tau) \frac{L_t}{G_t \gamma_t}, \\
P_{t,\tau}^{Baa} &= \alpha_1^{Baa}(\tau) + \alpha_2^{Baa}(\tau) \frac{1}{G_t \gamma_t} + \alpha_3^{Baa}(\tau) L_t + \alpha_4^{Baa}(\tau) \frac{L_t}{G_t \gamma_t},
\end{align*}
\]

where we omit the superscript "RFV" for simplicity. Since $SP^{BA} = \frac{1}{\tau} \log \left( \frac{P_{t,\tau}^{Baa}}{P_{t,\tau}^{Aaa}} \right)$ and using the fact that $P_{t,\tau}^{Baa}/P_{t,\tau}^{Aaa}$ is close to one, we can approximate $SP^{BA}$ by

\[
SP^{BA} \approx \frac{1}{\tau} \left( \frac{1}{\alpha_1^{Baa}(\tau) + \alpha_2^{Baa}(\tau) \frac{1}{G_t \gamma_t} + \alpha_3^{Baa}(\tau) L_t + \alpha_4^{Baa}(\tau) \frac{L_t}{G_t \gamma_t}} \right) \left( \frac{\left[ \alpha_1^{Aaa}(\tau) - \alpha_1^{Baa}(\tau) \right] + \left[ \alpha_2^{Aaa}(\tau) - \alpha_2^{Baa}(\tau) \right] \frac{1}{G_t \gamma_t}}{\left[ \alpha_3^{Aaa}(\tau) - \alpha_3^{Baa}(\tau) \right] L_t + \left[ \alpha_4^{Aaa}(\tau) - \alpha_4^{Baa}(\tau) \right] \frac{L_t}{G_t \gamma_t}} \right). 
\]

(A.61)

From (A.61), the average of spreads are mainly driven by the four coefficient differences, while their standard deviation is mainly determined by $\alpha_j^{Aaa}(\tau) - \alpha_j^{Baa}(\tau)$ for
\( j = 2, 3, 4 \), which controls the sensitivity of \( SP^{BA} \) to \( \gamma_t \) and \( L_t \). Figure A.2 plots \( \alpha_j (\tau) \) as the function of the (four-year) cumulative default rate taking values from zero to the default rate for Baa firms.

Several properties about the pricing coefficients \( \alpha \) are in line. First, \( \alpha_2 (\tau) \) is positive but \( \alpha_3 (\tau) \) is negative implying that high risk aversion and high default loss both tend to drive down defaultable bond prices. For the interaction term in (A.61), \( \alpha_4 (\tau) \) is also positive implying that a high risk aversion drives down bond price more than a high loss rate. Second, \( \alpha_{1,b,Aaa} (\tau) > \alpha_{1,b,Baa} (\tau) \) implying less discount for Aaa bonds as the safer asset. This is the main driving force that leads to the positive \( SP^{BA} \). Third, in absolute values, \( \alpha_j (\tau) (j = 2, 3, 4) \) are larger for Baa bonds than for Aaa bonds implying that riskier assets are more sensitive to state variations than safer assets. This property drives the standard deviation of \( SP^{BA} \).

References


dynamic relationship between investment-grade bonds and credit default swaps.

volatility for credit default swap valuation. Journal of Financial Markets 13,
321-343.

explanation of aggregate stock market behavior, Journal of Political Economy
107, 205–251.


[14] Chen, H., 2010, Macroeconomic conditions and the puzzles of credit spreads and

credit spread puzzles and the equity premium puzzle, Review of Financial Studies
22, 3367–3409.


[17] Chen, N and S. G. Kou, 2009, Credit Spreads, Optimal capital structure, and
implied volatility with endogenous default and jump risk, Mathematical Finance
19, 343-378.


[20] Covitz, D., and C. Downing, 2007, Liquidity or credit risk? The determinants of

credit spreads: option-implied jump risk premia in a firm value model. Review
of Financial Studies 21, 2209–42.

University.


Figure 1. Bond risk premiums and credit spreads. Figure 1 plots bond risk premiums for defaultable bonds under independent defaults and disaster-triggered defaults as well as default-free bond risk premium. All plots are with respect to changes in economic conditions as proxied by the surplus ratio $S$. We keep the loss $L_t$ constant at its empirical average $\bar{L}$ in all the subplots. The top left panel plots the three jump risk premiums. The top right panel illustrates the default risk premiums. The bottom two panels plot the premium differences between defaultable and default-free bonds $BP_t^{ind} - BP_t^{0}$, together with the implied four-year Baa-treasury credit spreads under both independent default (left) and disaster-triggered default (right).
Figure 2. Impacts of countercyclical $\gamma$ and $L$ on 4-years Baa-treasury spreads. Figure 2 illustrates comparative statics concerning countercyclical default loss and time varying risk aversion on bond premiums and total spreads. This exercise is performed for 4-year Baa bonds. The top two panels plot bond risk premiums and their components under both the full model and the special case with constant risk aversion $\gamma$ fixed at $\bar{\gamma}$. The top left panel plots together the diffusive risk premium with and without variations in $\gamma$. The top right panel plots together the sum of the jump risk premium and the default risk premium. The bottom left panel reports the total spread in both these later cases. The bottom right panel plots total 4-year spread under both the full model and the special case with constant Loss rate $L$ fixed at $\bar{L}$. 
Figure 3. Decompositions of 5-years Baa-treasury spreads and 5-years Aaa-treasury spreads (Recovery of Face Value RFV arrangement). Figure 3 plots the implied 5-year Baa-treasury and 5-year Aaa-treasury spreads under five cases. In model 1, we assume independent defaults and set \( k = \alpha = k_L = \alpha_L = 0 \). In model 2, we allow disaster-triggered defaults but still set \( k = \alpha = k_L = \alpha_L = 0 \). In model 3, we allow disaster-triggered defaults and only permit variations in \( L \) by setting \( k = \alpha = 0 \). In model 4, we allow disaster-triggered defaults and only permit variations in \( \gamma \) by setting \( k_L = \alpha_L = 0 \). Model 5 is the full model. The top two panels plot the total spread for 5-year Baa-treasury spread both as a function of surplus ratio \( S \) (left panel) and as a function of initial loss rate (right panel). The bottom two panels repeat this exercise for the 5-year Aaa-treasury spreads.
Figure 4. State dependences of four year Baa-Aaa spreads. Figure 4 plots the 4-year Baa-treasury spreads implied from our full model for different levels of loss rates and risk aversions.
**Figure 5. Term structure of Baa-Aaa spreads.** Figure 5 plots the term structure of Baa-Aaa credit spreads for different maturities. We calibrate our full model to the term structure of objective default probabilities. Cumulative default probabilities for maturities ranging from 1 to 20 years for Baa and Aaa bonds are from Moody’s handbook. In particular, we use two default rate averages: 1970–2008 and 1920–2008. The corresponding model implications for maturities up to 10 years are plotted in the top left and the top right panel of Figure 5, respectively. The bottom two panels of the same figure do this exercise for maturities up to 20 years. We split both range of maturities to help the reader focus on each part of the curve at one time.
Figure 6. Time series of credit spreads. We obtain the innovation of the historical log consumption growths from St Louis Fed and Campbell Shiller’s website. We use this innovation to construct the time series of the surplus ratio and loss rate using (2.1)–(2.2) and (2.12), respectively. Given the time series of $S$ and $L$, we back out the model implied spreads and compare them to historical data. The top two panels of Figure 6 plot together the model-implied credit spreads and their changes vs the historical values. The bottom two panels plot the same model implications vs their data counterparts when we shut down variations in the loss rate $L$. 
Figure 7. Credit spread vs. smirk premium: comparative analysis with respect to the absolute consumption jump size. Top two panels of Figure 7 plot the implied average 4-years Baa-Aaa credit spread and average smirk premiums measured as the implied volatility difference between 10% OTM puts and ATMs. We do this exercise as we vary the absolute log consumption jump size $|b|$. To give a more direct comparison, the bottom left panel of the same figure plot the average credit spread against the average smirk premium as we vary $|b|$. The bottom right panel also plots the average credit spread against the average equity premium as we vary $|b|$.
Figure 8. Credit spread, smirk premium, and bond premium difference: The top two panels plot the state dependences of the 4-year Baa-Aaa spreads and the option smirk premiums, where the default loss is fixed at its long run average. The smirk premium is measured as the implied volatility difference between 10% OTM puts and ATMs. We decompose credit spreads into its own two big components: expected loss versus bond risk premium. The bottom two panels plot premium difference between Baa and Aaa bonds against the smirk premium under both the low and the high high levels of the surplus ratio $S$. 

---

60
Figure 9. Robustness. The top two panels plot the sensitivity of the average and standard deviation of 4-year Baa-Aaa spreads to the three model parameters controlling, respectively, the sensitivity of habit and loss to consumption shock ($\alpha$ and $\alpha_L$), and the absolute consumption jump size ($|b|$). The bottom two panels plot the 5-year Baa-treasury spreads and the implied ratio of default intensity between $P$ and $Q$ measure against idiosyncratic defaults as the fraction of the total default. In the bottom left panel, we also plot the observed average level of 5-year CDS spreads for Baa rated firms which is indicated by the dotted horizontal line.
Table I.
The base case calibration and the pricing of equity and options

Panel A&B report the base case calibration of our model. Panel A reports calibration of consumption process (excluding the disaster component) and the preference parameters; panel B reports calibration of parameters related to disasters defaults, where \( L \) is the loss rate; \( \lambda^{D,Aaa} \) and \( \lambda^{D,Baa} \) are the annualized default intensities for 4-year Aaa and Baa bonds, respectively. Panels C&D report the pricing of aggregate equity and equity index options, respectively. In particular, we report implications from both our model and their empirical values documented in the literature.

<table>
<thead>
<tr>
<th>Panel A: parameters not related to disasters and defaults</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu ) (%)</td>
</tr>
<tr>
<td>2.05</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: parameters related to disasters and defaults</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b ) = (-b)</td>
</tr>
<tr>
<td>0.022</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C: moments about short term rate and aggregate equity (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( mean(r) )</td>
</tr>
<tr>
<td>model values</td>
</tr>
<tr>
<td>empirical values</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel D: part of the option pricing moments (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>moneyness</td>
</tr>
<tr>
<td>model values</td>
</tr>
<tr>
<td>empirical values</td>
</tr>
</tbody>
</table>
Table II.

Pricing of defaultable bonds

Table II reports the pricing of defaultable bonds in terms of the four year Aaa and Baa bonds under the RFV arrangement, i.e., bondholders recover a fraction of the bond face value upon bankruptcy. Panel A of Table II reports the 4-year Baa-Aaa spreads under five different models. In model 1, we assume independent defaults and set \( k = \alpha = k_L = \alpha_L = 0 \). In this model, bond risk premium is zero, and credit spread is only attributed to the default loss at the average level \( \bar{L} \). In model 2, we allow disaster-triggered defaults but still set \( k = \alpha = k_L = \alpha_L = 0 \). In this model, channel 1, i.e., the positive correlation between pricing kernel and default time, is open, but we shut down variations in both states. In model 3, we allow disaster-triggered defaults and only permits variations in \( L \) by setting \( k = \alpha = 0 \). In model 4, we allow disaster-triggered defaults and only allows variations in \( \gamma \) by setting \( k_L = \alpha_L = 0 \). Model 5 is our full model. Panel B of Table II reports the state-dependences of the default intensity ratio \( \frac{\lambda Q}{\lambda P} \).

<table>
<thead>
<tr>
<th>Panel A: Baa-Aaa spreads under different models</th>
<th>mean (bp)</th>
<th>std. deviation (bp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>default loss</td>
<td>18.4</td>
<td>0</td>
</tr>
<tr>
<td>no state variations</td>
<td>21.8</td>
<td>0</td>
</tr>
<tr>
<td>only variations in ( L )</td>
<td>39.5</td>
<td>1.9</td>
</tr>
<tr>
<td>only variations in ( \gamma )</td>
<td>52.0</td>
<td>12.4</td>
</tr>
<tr>
<td>full model</td>
<td>112.7</td>
<td>46.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Default intensity ratio</th>
<th>surplus ratio</th>
<th>default intensity ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0191</td>
<td>6.17</td>
<td></td>
</tr>
<tr>
<td>0.0262</td>
<td>5.02</td>
<td></td>
</tr>
<tr>
<td>0.0333</td>
<td>3.87</td>
<td></td>
</tr>
<tr>
<td>0.0381</td>
<td>3.10</td>
<td></td>
</tr>
<tr>
<td>0.0429</td>
<td>2.34</td>
<td></td>
</tr>
<tr>
<td>0.0453</td>
<td>1.95</td>
<td></td>
</tr>
<tr>
<td>0.0176</td>
<td>1.57</td>
<td></td>
</tr>
<tr>
<td>average</td>
<td>3.09</td>
<td></td>
</tr>
</tbody>
</table>
Figure A.1. Decompositions of 5 years Baa-treasury and Aaa-treasury: TP arrangement. Figure A.1 plots the implied 5y Baa-treasury and 5y Aaa-treasury spreads under five cases. In model 1, we assume independent defaults and set $k = \alpha = k_L = \alpha_L = 0$. In model 2, we allow disaster-triggered defaults but still set $k = \alpha = k_L = \alpha_L = 0$. In model 3, we allow disaster-triggered defaults and only permits variations in $L$ by setting $k = \alpha = 0$. In model 4, we allow disaster-triggered defaults and only permits variations in $\gamma$ by setting $k_L = \alpha_L = 0$. Model 5 is our full model. The top panels plot the total spread for 5-year Baa-treasury spread both as a function of surplus ratio $S$ (left panel) and as a function of initial loss rate (right panel). The bottom panels repeat this exercise for the 5-year Aaa-Treasury spreads.
Figure A.2. Pricing coefficients in response to changes in default probabilities. Conditional on no default, price of defaultable bonds with \( \tau \)-period to expiration is given by

\[
P_{t,\tau}^{bIP} = \alpha_1^{IP}(\tau) + \alpha_2^{IP}(\tau) \frac{1}{\gamma_t} + \alpha_3^{IP}(\tau) L_t + \alpha_4^{IP}(\tau) \frac{L_t}{\gamma_t},
\]

where \( \gamma_t \) and \( L_t \) are the two states. Figure A.2 plots the pricing coefficient \( \alpha_j^{b}(\tau) \) for \( j = 1, 2, 3, 4 \) against the cumulative default probabilities for four-year bonds.