# Quantity Competition in C2C Exchange Markets with Strategic Consumers and Dynamic Preferences

Sam Kirshner
UNSW Business School, University of New South Wales, Sydney, Australia

Yuri Levin and Mikhail Nediak
Smith School of Business, Queen's University, Kingston, Canada

**Abstract**

This paper develops a methodology for studying quantity competition across an assortment of substitute products in a C2C exchange market. Consumers are differentiated by their preferences for products and maximize surplus given uncertainty in their future preferences. The equilibrium is characterized by a unique price path, which is calculated by recursively solving a series of linear complementarity problems. We explore the interaction between strategic behavior and consumer preference dynamics. Our analysis indicates that strategic behavior incentivizes purchasing behavior based on resale value rather than utility, decreasing market surplus. However, we find that increased uncertainty and homogeneity in preference dynamics impacts prices, which counteracts the negative influence of strategic behavior. In addition to enabling firms to study market surplus, the model offers C2C platform operators a price recommendation tool to increase market efficiency.

## 1. Introduction

With the growth of e-commerce, consumer-to-consumer (C2C) markets have become an increasingly important platform for consumers to purchase and sell used products. The annual estimated value of goods traded in C2C markets exceeds \$190 billion, and is expected to exceed \$420 billion by 2020 (Dobbs et al. 2013). Until recently, a few established firms such as eBay, Craigslist, and Taobao have dominated C2C exchange markets. However, low barriers to entry and the increasing popularity of mobile two-sided markets have led to a substantial growth in mobile platforms for selling used-goods. New platforms are appealing to consumer trends in sustainability by highlighting the environmental benefits of reducing production and landfill waste through exchanges. Given the increase in volume and value of transactions as well as the potential impacts on sustainability, its increasingly important to understand the factors that influence consumer surplus in established and emerging virtual C2C markets.

Although literature has analyzed consumer surplus in B2C quantity competition models (see Kluberg and Perakis (2012) for a recent discussion), there is an absence of analogous studies for C2C markets. Existing models of quantity competition are unsuited for the study of C2C markets because firm objectives and decisions are inherently different from C2C participants. Firms operating in B2C markets are profit-maximizers deciding on the quantity to supply the market in a competitive setting. In comparison, C2C participants are surplus-maximizing agents who decide the quantity of products to either buy or sell. Thus, prices in B2C markets determine the quantity that firms are willing to supply, whereas prices in C2C markets determine whether consumers, based on their utility and ownership of products, act as buyers or sellers. As a result, the market clearing prices will have a different structure compared to similar markets where consumers purchase directly from firms.

To study quantity competition in a used-good C2C marketplace, we develop a novel approach to explicitly model consumer behavior using multiple representative consumers. Each representative consumer characterizes a portion of the population with a similar preference profile. By considering the size of each consumer segment and the value placed on products, we construct a quadratic utility function for each representative consumer. The representative consumers compete in a model of quantity competition, where a market clearing mechanism determines the equilibrium prices. Although markets are rarely in exact equilibrium in practice, the approach approximates the behavior of consumers in C2C markets.

The quadratic utility functions and consumer supply constraints reduce to a linear complementarity problem (LCP) describing demand. Aggregating each representative consumer's demand function creates a market LCP. We show that finding a solution to the market LCP is equivalent to finding a set of market clearing prices. Using the properties of the LCP and the structure of the aggregate market preferences, we establish that the set of market clearing prices and the equilibrium quantity of products traded is unique. Equilibrium analysis reveals that the unique set of prices is determined by the aggregate preference structure and the total supply of products. Thus, prices and the resulting aggregate surplus of the market are independent of the initial ownership of products across the market. Furthermore, we show that the aggregate surplus is not necessarily increasing in the number of participants and supply of products. This result contrasts B2C quantity competition, where total supply and thus consumer surplus is nondecreasing in the number of firms supplying overlapping product sets.

We refer to this model as the *static C2C market* because pricing and consumer decisions are only based on the current period. The model is applicable to C2C markets encompassing a wide variety of perishable or semi-durable goods where consumers purchase purely for consumption and do not consider resale value of the product. Examples of these types of products include concert tickets, textbooks, and clothing. The static C2C market is also relevant for studying applications in the sharing economy. Unlike firms that operate in rental markets, owners providing goods or services in the sharing economy have utility for their products. For example, it is estimated that 87% of the 15,000 AirBnB hosts in New York City rent the homes that they live in and that half of these hosts rely on income generated by AirBnB to keep their homes (Chesky 2013). As a result, participants in the sharing economy face a trade-off between utility and revenue for sharing the product or service.

Consumer valuations for durable products may vary over time due to changes in taste or financial circumstances. For example, consumer preferences for cars, houses, and collectibles tend to change with age. For products of considerable value, consumers may act strategically in terms of product selection and purchase timing. As a result, the equilibrium prices from the static model will not hold when willingness to pay accounts for future preferences and the potential resale value of the products.

We extend the static C2C model to a multi-period setting, to examine markets where consumers are strategic and experience inter-temporal changes in their product preferences and price sensitivities. The multi-period model, which we refer to as the *dynamic C2C market*, uses our multiple representative consumer approach to capture changes in preferences. The utility parameters of a representative consumer are independent of the population size and represents a "type" of preference profile. At the end of each period, the individual consumers that comprise a representative consumer changes types with a certain probability. Thus, the population size of each representative consumer is dynamic and individual consumers face uncertainty in their future preference profile.

We show that a unique price path characterizes the equilibrium and that it can be calculated by recursively solving a series of LCPs. The unique price path leads to a unique set of exchanges amongst the market participants that is independent of the initial allocation of products. Under market scenarios where representative consumers do not sell their entire supply of products, equilibrium prices and quantities traded can be expressed in closed-form based entirely on input parameters.

The equilibrium is influenced by the several factors related to the preference dynamics of representative consumers. Specifically, the population size and price sensitivity of each representative consumer together with their level of certainty of future preferences create a weighted average that we refer to as the level of *market certainty*. Individual consumers may have greater (less) certainty compared to the market level, in which case we refer the consumer as having *excess certainty* (*excess uncertainty*).

The optimal market-clearing prices are comprised of a current value and future value component. The current value component is the set of prices generated by the static C2C model. The future value component is dependent on the expected future prices, the level of strategic behavior, and the market certainty. Consumers with individual excess certainty purchase products that have greater value in the following period. We refer to this prospective behavior as the *resale effect*. The resale effect decreases aggregate surplus because (1) consumers with excess certainty purchase products that may not match their preference profile and (2) higher prices resulting from the resale effect deter consumers with excess uncertainty from purchasing products that match their preference profile. Greater uncertainty in consumer dynamics causes prices to tend towards the static C2C market's price behavior, while greater homogeneity in dynamics decreases the resale effect. Thus, greater uncertainty and homogeneity diminish the adverse effects of strategic behavior on aggregate surplus.

Finally, we extend the framework to a special case of an infinite horizon setting where population dynamics are stationary. We find that the structure of the prices and the interpretation of the results when consumer dynamics are stationary in the infinite horizon setting is consistent with the results in the two period setting. This infinite horizon setting demonstrates that consumers' levels of certainty influence the prices and consumer surplus, even when the aggregate preferences for products are stationary.

The proposed methodology provides firms operating C2C markets with a decision support tool for making pricing recommendations to maximize market surplus. Unlike firms that have extensive resources to gather or purchase data to make informed pricing decisions, C2C participants may have limited knowledge of the demand structure or product supply of other participants. As a result, consumers' rationality is bounded and they may make products available at prices that do not necessarily clear the market. Given the experimental and empirical support for reference price and anchoring effects, price recommendations can help induce stable markets, where consumers make exchanges that maximize surplus.

Although the literature has suggested that marketplaces should coordinate participation of users by helping to formulate effective pricing structures (Sriram et al. 2013; Einav et al. 2015), few C2C marketplaces provide recommendations to users. The benefit of suggesting prices can be seen by AirBnB's new Price Tips tool for apartment hosts. Since introducing Price Tips, hosts that price within 5% of the recommended price are nearly four times as likely to make a transaction (Huet 2015). A pricing scheme that helps users account for preferences, the supply, and substitution effects between products will ultimately lead to more transactions and greater value for participants.

The methodology is also relevant for studying collectibles, where preference changes (due to changes in both wealth and taste) and strategic behavior are inherent characteristics of market participants. For many collectible categories such as antique furniture, collectors have little insight into the value of a product from professional dealers or its prospects for appreciation. This lack of knowledge can create significant distortions in prices, which result in higher levels of buyer and seller remorse. Distortions in valuing products for the future is likely to be even more egregious when present valuations are uncertain. Consequently, unlike markets for most other types of goods, increased buyer participation in C2C markets for collectibles generally leads to decreased price efficiency and less aggregate surplus (Kauffman et al. 2009).

Recently, Bapna et al. (2008) proposed a model for measuring consumer surplus in auctions using data from eBay. However, C2C trading has shifted dramatically from auctions to posted prices. In addition to the growth in mobile C2C markets which use listed prices or prices generated by the platform, 90% of items exchanged on eBay use posted prices rather than auctions (Einav et al. 2013). In view of the increasing scale of these markets, there is a need for a general model that can help researchers studying markets to assess the impact of transactions on the surplus of participants.

## 1.1 Related Literature

In operations management, used-goods C2C markets have traditionally been studied in the presence of a retailer selling a durable product. As a result, research questions have focused on a variety of operational decisions from the perspective of the retailer in situations where consumers have access to a secondary market. Specific examples include the decision between leasing and selling (Desai and Purohit 1998; Huang et al. 2001; Bhaskaran and Gilbert 2005), the choice of product durability (Hendel and Lizzeri 1999; Johnson 2011), product upgrade strategies (Yin et al. 2010), re-licensing strategies (Oraiopoulos et al. 2012), and structuring of product return contracts (Gümüs et al. 2013). The majority of this literature considers stylized models based on a single type of product with consumers who are differentiated by their valuation for that product. In this paper, we consider the assortment of durable used-good products rather than a single product type in a general C2C setting. The setup of the model allows products to be differentiated by brand, vintage, products condition, and taste/fit features such as color and size. Although we do not focus on the impact of secondary markets on the operational decisions of the firm, the proposed approach can be utilized to extend existing models to a multi-product setting.

In the oligopoly literature, demand for the differentiated products is often modeled using a representative consumer who has a quadratic utility function. Quadratic utility is analytically tractable and often sufficient to guarantee a unique outcome for the oligopoly equilibrium with limited assumptions on the utility parameters (see Farahat and Perakis (2011); Adida and DeMiguel (2011); Kluberg and Perakis (2012)). We follow the quadratic utility approach; however, due to the nature of C2C exchanges, we consider multiple representative consumers. The quadratic utility assumption leads to linear demand for each consumer segment, where negative demand implies that the consumer would rather sell than purchase the product. Although quadratic utility and linear demand are not valid for all consumers in a market (Kim et al. 2002), recent research has shown that linear demand is effective at approximating unknown demand curves given the intuitive condition that demand is nonincreasing in price (Besbes and Zeevi 2015; Cohen et al. 2015).

As Kachani and Shmatov (2011) discuss, modeling multi-product oligopolies is complicated by the interdependence between product demand and prices, in addition to differentiation amongst products produced by competing firms. For example, linear demand may lead to regions of prices that yield negative demand values for products, while potentially inflating the demand for a substitute to an arbitrarily high level. Soon et al. (2009) demonstrate that reformulating the multi-product linear demand as a LCP allows the demand function to be properly constructed over the entire set of non-negative prices. This approach was utilized by Farahat and Perakis (2010), who reduce the LCP to a linear program, and by Federgruen and Hu (2015) to study a price competition model with an arbitrary number of competitors each offering a product line of substitutable products. Federgruen and Hu (2016) generalize their previous model to characterize the equilibrium price behavior across a supply chain network.

In the exchange market setting, competitive linear demand functions lead to regions of prices where consumers want to sell beyond their product endowment. Thus, to ensure that the goods supplied in equilibrium do not exceed the quantity owned by the sellers, our model utilizes a LCP formulation similar to Soon et al. (2009), Farahat and Perakis (2010), and Federgruen and Hu (2015). Unfortunately, the quadratic utility is insufficient to guarantee uniqueness or even the existence of an equilibrium solution. A similar complication arises in Federgruen and Hu (2015), due to the presence of multiple price-competing retailers. Focusing on consumer segments that have correlated price sensitivities, we are able to show that the equilibrium quantities traded and the market-clearing price are unique. The assumption that consumers have correlated price sensitivities implies that they react similarly to changes in prices on a product level, but the scale of the reaction depends on the wealth of participants.

The model and analysis presented also complements the growing literature on strategic consumer behavior in revenue management. A recent review by Shen and Su (2007) classifies strategic customer behavior research into two groups, papers that examine the effect of customer inter-temporal substitution and papers pertaining to customer choice in multi-product revenue management settings. In our paper, consumers consider both inter-temporal and multi-product substitution. Consumers make strategic decisions given their uncertainty of future product needs and expectations of the price path. Since purchases lead to transfers of ownership of inventory and the dynamics can create either surplus or scarcity of product demand, the model allows for consumers to strategically time their purchases in anticipation of either price-drops or price increases. In this sense our paper is related to behavioral models of firm dynamic pricing and capacity availability decisions for firms selling perishable products.

Although products in this model are durable and provide tangible benefits to consumers in each period, the time-dependent nature of preferences implies that the value of a product can be perishable. As a result many results stemming from counteracting strategic behavior and oligopolistic competition in the presence of strategic consumers hold in the context of a non-cooperative exchange market. For example, similar to Su (2007), who studies pricing in a heterogeneous strategic consumer environment, we demonstrate that the level of strategic behavior and the valuations of consumers determine if the optimal price path is increasing, decreasing, or non-monotone with time. In addition, the results on strategic behavior support findings in Bazhanov et al. (2015), who study quantity competition with strategic consumers. They show that consumers may not benefit from being more strategic, because it prompts the suppliers to reduce the capacity leading to greater competition for a smaller quantity of a product. The impact of strategic behavior in C2C markets produces a stronger effect, since consumers with excess certainty over future preferences engage in more intense competition for a fixed set of valuable products, decreasing surplus.

### 1.2 Organization

The remaining part of the paper is organized as follows. Section 2 explains the behavior of consumers and introduces the notion of multiple representative consumers. Section 3 develops a one period C2C exchange market and establishes uniqueness in terms of the products traded and the market prices. We extend the static model to introduce consumer dynamics of preferences in section 4. After describing modeling dynamics, we utilize the discussion and results from the static model to show that uniqueness of the quantities exchanged and prices hold in a multi-period setting. We focus on how strategic behavior interacts with consumer dynamics to influence prices, quantities, and surplus. In section 5, we extend several of the results from section 4 to a stationary infinite horizon setting. The paper concludes with a brief summary and discussion of future research in Section 6. All proofs can be found in the Technical Appendix.

## 2. Characterizing Consumers

Consider a population of $m$ types of heterogeneous consumers, where each segment $i \in \mathcal{I} \equiv \{1 \dots m\}$ has size $k_i$. Types of consumer segments are characterized by the preference set of products and as well as sensitivity to changes in prices and is independent of the population size and time. There are $n$ types of durable heterogeneous products, which are gross-substitutes, owned by members of the population. The total number of products $j \in \mathcal{J} \equiv \{1, ..., n\}$ owned by a population group is denoted by the vector $\mathbf{q}_i$. The combination of $\mathbf{q}_i$ and utility parameters $\tilde{\mathbf{a}}_i$ and $\tilde{\mathbf{B}}_i$ define the utility of consumer group $i$. The vector $\tilde{\mathbf{a}}_i = ||\tilde{a}_{ij}||$ is interpreted as a person of type $i$ willingness to pay for each product $j$ provided that the customer owns no products. The matrix $\tilde{\mathbf{B}}_i = ||\tilde{b}_{ijj'}||$, is the decrease in the willingness to pay for products $j$ as quantity of good $j'$ increases.

To develop the intuition behind our approach, we start by examining the utility earned by the typical individual within a consumer group $i$. The value provided by each product is assumed to be a linear function dependent on the type and quantity of other products owned by the average consumer. If the total number of products owned by the group is $\mathbf{q}_i$, then the average consumer owns $\mathbf{q}_i/k_i$. Given the definitions of $\tilde{a}_{ij}$ and $\tilde{\mathbf{B}}_{ij}$, the $j^{th}$ row of matrix $\tilde{\mathbf{B}}_i$, the value provided by product $j$ to a type $i$ consumer endowed with quantity $\mathbf{q}_i/k_i$ is $\tilde{a}_{ij} - \frac{1}{2}\tilde{\mathbf{B}}_{ij}\mathbf{q}_i/k_i$. Since the average consumer owns quantity $\mathbf{q}_i/k_i$, the individual utility for the average consumer $i$ is

$$\tilde{u}_i(\mathbf{q}_i) = \left(\frac{\mathbf{q}_i}{k_i}\right)^T \left(\tilde{\mathbf{a}}_i - \frac{1}{2k_i}\tilde{\mathbf{B}}_i\mathbf{q}_i\right). \tag{2.1}$$

To calculate the total utility for the consumer segment $i$, the individual utility (2.1) is multiplied by the population size. The total utility of the representative consumer group $i$ owning product quantities $\mathbf{q}_i$ is the quadratic expression

$$u_i(\mathbf{q}_i) = \mathbf{q}_i^T \left( \tilde{\mathbf{a}}_i - \frac{1}{2k_i} \tilde{\mathbf{B}}_i \mathbf{q}_i \right)$$

Each representative consumer's willingness to pay is assumed to be positive for all products. Since we are considering substitutable products, $\tilde{\mathbf{B}}_i$ is assumed to be positive definite for all $i$. The assumption on $\tilde{\mathbf{B}}_i$ implies that the utility function is strictly concave for each representative consumer.

Each consumer group can be equivalently characterized by the parameters $\mathbf{B}_i = \tilde{\mathbf{B}}_i^{-1}$ and $\mathbf{a}_i = \mathbf{B}_i \tilde{\mathbf{a}}_i$. The vector $\mathbf{a}_i$ is the *ideal quantity* set of consumer $i$, independent of the price. The matrix $\mathbf{B}_i$ is the demand sensitivities to changes in price. Notice that the vector $\mathbf{B}_i \tilde{\mathbf{a}}_i$ is the per person demand for products and $\mathbf{B}_i$ is the per person price elasticity of demand. Both of these values are independent of the population size of the group, implying that a characterization $i$ is uniquely defined by individual product preference and sensitivities. Since, $\tilde{\mathbf{B}}_i$ is positive definite, the diagonals of $\mathbf{B}_i$ are strictly positive. Matrices $\mathbf{B}_i$ are also assumed to be strictly row and column diagonally dominant, i.e. $b_{ijj} > \sum_{j \neq j'} |b_{ijj'}|$ and $b_{ijj} > \sum_{j \neq j'} |b_{ij'j}|$ for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$. As Federgruen and Hu (2015) discuss, the literature modeling gross substitute markets often employ this assumption, since it is intuitive and likely to hold in most applications.

For notational convenience, throughout the paper, parameters subscripted by $i$ with single bar accents account for the size of population $i$. For example, $\bar{\mathbf{a}}_i = k_i \mathbf{a}_i$ and $\bar{\mathbf{B}}_i = k_i \mathbf{B}_i$ correspond to the ideal set of products owned and price sensitivities of representative consumer $i$ with population size $k_i$. Parameters with double bar accents are used to replace the sum of parameters indexed over the set $\mathcal{I}$. For example, the terms $\bar{\bar{\mathbf{a}}} = \sum_{i \in \mathcal{I}} \bar{\mathbf{a}}_i$ and $\bar{\bar{\mathbf{B}}} = \sum_{i \in \mathcal{I}} \bar{\mathbf{B}}_i$ represents the overall markets ideal product set and price sensitivity.

# 3. Static C2C Market

To develop the intuition for the model, we begin by separating the decisions of the representative consumer to buy and sell products within the C2C market. Define the variables $\mathbf{y}_i^s$ as the number of units of each products available for trade by customer group $i$ and $\mathbf{y}_i^d$ as the demand for each product by customer group $i$ given a price vector $\mathbf{p}$. If representative consumer $i$ starts with $\mathbf{x}_i > 0$ quantities then the quantities owned after the participating in the exchange market is $\mathbf{q}_i = \mathbf{x}_i + \mathbf{y}_i^d - \mathbf{y}_i^s$. Thus, the surplus earned by representative consumer $i$ after transactions is

$$v_i = \left(\mathbf{x}_i + \mathbf{y}_i^d - \mathbf{y}_i^s\right)^T \left( \tilde{\mathbf{a}}_i \quad - \tfrac{1}{2k_i} \tilde{\mathbf{B}}_i \left(\mathbf{x}_i + \mathbf{y}_i^d - \mathbf{y}_i^s\right) \right) - (\mathbf{y}_i^d - \mathbf{y}_i^s)^T \mathbf{p}. \tag{3.1}$$

Supply is restricted by the constraints $\mathbf{0} \leq \mathbf{y}_i^s \leq \mathbf{x}_i$, since consumers cannot sell more products than they own. Similarly, demand is nonnegative, so the representative consumer decisions are constrained by $\mathbf{0} \leq \mathbf{y}_i^d$. Under the assumption that there is no opportunity for arbitrage, no consumer will buy and sell the same product in equilibrium. This implies that the complementarity condition $\mathbf{0} \leq \mathbf{y}_i^s \perp \mathbf{y}_i^d \geq \mathbf{0}$ will always be satisfied in equilibrium. Thus, for simplicity we let the variable $\mathbf{y}_i = \mathbf{y}_i^d - \mathbf{y}_i^s$ be the joint supply and demand of consumer $i$.

Each representative consumer consists of many individuals. Individual consumers do not have the ability to influence or set prices that would be sustained in equilibrium. Thus, consumers are price takers and the market reaches an equilibrium at a price that clears the market. For ease of exposition, we introduce a fictitious agent into the market. The fictitious agent's objective is to set price such that market clearing condition $\sum_{i\in\mathcal{I}} \mathbf{y}_i = 0$ is satisfied. If the fictitious agent selects prices $\mathbf{p}$, then each consumer groups is trying to maximize their surplus

$$\begin{aligned} v_i(\mathbf{x}_i) = \max_{\mathbf{y}_i} \quad & \big(\mathbf{x}_i + \mathbf{y}_i\big)^T \left( \tilde{\mathbf{a}}_i - \frac{1}{2k_i} \tilde{\mathbf{B}}_i \big(\mathbf{x}_i + \mathbf{y}_i\big) \right) - \mathbf{y}_i^T \mathbf{p} \\ \text{s.t.} \quad & \mathbf{x}_i + \mathbf{y}_i \geq \mathbf{0}. \end{aligned}$$

**Proposition 1.** *For price $\mathbf{p}$, the optimal quantity decisions for consumer segment $i$ is uniquely determined by the complementarity problem*

$$\mathbf{0} \leq \mathbf{s}_i \perp \mathbf{x}_i + \mathbf{y}_i = \bar{\mathbf{a}}_i - \bar{\mathbf{B}}_i \big(\mathbf{p} - \mathbf{s}_i\big) \geq \mathbf{0}. \tag{3.2}$$

**Corollary 1.** *The supply of product $j$ by consumer $i$ is increasing in price, quantities of product held, and product preference. The demand for product $j$ by consumer $i$ is decreasing in price, quantities of product held, and increasing in product preference.*

Given a set of prices for each product, a consumer will want to potentially sell more goods than is owned. The multipliers $\mathbf{s}_i$ act as a correction term on the price, to create a set of prices $\mathbf{p}'_i = \mathbf{p} - \mathbf{s}_i$, such that the consumer's supply constraint is satisfied. This is similar to the demand function of a representative consumer facing multiple retailers engaged in competition, where the multipliers act to correct the demand for products, such that no consumers have non-negative demand for any individual (see Kluberg and Perakis 2012; Federgruen and Hu 2015). While a positive value for a multiplier in oligopolistic competition implies that the corresponding product has zero demand, in the exchange market active multipliers imply that the consumer wants to sell more products than available and is being constrained by the number of products owned. If $\mathbf{p}$ satisfies the market clearing conditions, than in equilibrium, an active multiplier implies that there is sufficiently high demand for the consumer's product from other consumer segments.

Incorporating the fictitious agent's market clearing condition, an equilibrium solution of products traded can be found by solving the complementarity problem:

$$\bar{\mathbf{a}}_i - \bar{\mathbf{B}}_i \big(\mathbf{p} - \mathbf{s}_i\big) - \mathbf{x}_i - \mathbf{y}_i = \mathbf{0}, \quad \forall i \in \mathcal{I} \tag{3.3}$$

$$\mathbf{0} \leq \mathbf{s}_i \perp \mathbf{x}_i + \mathbf{y}_i \geq \mathbf{0}, \quad \forall i \in \mathcal{I} \tag{3.4}$$

$$\sum_{i\in\mathcal{I}} \mathbf{y}_i = \mathbf{0}. \tag{3.5}$$

We demonstrate the uniqueness of the solution to the exchange market for the case of correlated demand cross-elasticities. Defining $\mathbf{B}$ as the market demand sensitivity to prices, consumers are differentiated by their relative sensitivity parameter $\gamma_i > 0$, such that $\mathbf{B}_i = \gamma_i \mathbf{B}$ for all $i$. Consumer groups are further differentiated by their product preference $\mathbf{a}_i$, their population size $k_i$, and the existing products currently owned $\mathbf{x}_i$. The total available products in the market are $\bar{\bar{\mathbf{x}}} = \sum_{i\in\mathcal{I}} \mathbf{x}_i > \mathbf{0}$. With the correlated demand elasticities, $\bar{\mathbf{B}}_i = \bar{\gamma}_i \mathbf{B}$, where $\bar{\gamma}_i = \gamma_i k_i$. For convenience we introduce the normalization parameters $\hat{\gamma}_i = \bar{\gamma}_i / \bar{\bar{\gamma}}$, where $\bar{\bar{\gamma}} = \sum_{i\in\mathcal{I}} \bar{\gamma}_i$. To demonstrate uniqueness, we reformulate the mixed LCP (3.3)-(3.5) into a standard LCP. Define $\boldsymbol{\rho} = \bar{\bar{\mathbf{a}}} - \bar{\bar{\mathbf{x}}}$ and the block vectors

$$\mathbf{q} = \begin{bmatrix} \mathbf{x}_1 + \mathbf{y}_1 \\ \mathbf{x}_2 + \mathbf{y}_2 \\ \vdots \\ \mathbf{x}_m + \mathbf{y}_m \end{bmatrix} \qquad \text{and} \qquad \mathbf{r} = \begin{bmatrix} \bar{\mathbf{a}}_1 - \hat{\gamma}_1 \boldsymbol{\rho} \\ \bar{\mathbf{a}}_2 - \hat{\gamma}_2 \boldsymbol{\rho} \\ \vdots \\ \bar{\mathbf{a}}_m - \hat{\gamma}_m \boldsymbol{\rho} \end{bmatrix},$$

and the block matrix

$$\mathbf{M} = \begin{bmatrix} \bar{\gamma}_1(1-\hat{\gamma}_1)\mathbf{B} & -\bar{\gamma}_1\hat{\gamma}_2\mathbf{B} & \cdots & -\bar{\gamma}_1\hat{\gamma}_m\mathbf{B} \\ -\bar{\gamma}_2\hat{\gamma}_1\mathbf{B} & \bar{\gamma}_2(1-\hat{\gamma}_2)\mathbf{B} & \cdots & -\bar{\gamma}_2\hat{\gamma}_m\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\gamma}_m\hat{\gamma}_1\mathbf{B} & -\bar{\gamma}_m\hat{\gamma}_2\mathbf{B} & \cdots & \bar{\gamma}_m(1-\hat{\gamma}_m)\mathbf{B} \end{bmatrix}.$$

The blocks $\mathbf{q}_i = \mathbf{x}_i + \mathbf{y}_i$ in vector $\mathbf{q}$ are the quantities owned by each consumer after the consumers exchange goods. The vector $\boldsymbol{\rho}$, which we call the product imbalance, represents the total population's (across all consumer groups) ideal quantity set of each product relative to the total supply of the product. If a component of the product imbalance is positive (negative), then there is a shortage (surplus) of the product relative to aggregate ideal quantities. Thus, the blocks $\mathbf{r}_i = \bar{\mathbf{a}}_i - \hat{\gamma}_i \boldsymbol{\rho}$ in vector $\mathbf{r}$ represent the relative desire for each product accounting for the consumer's sensitivity to price and the product imbalance, but is independent of demand. If for product $j$, the imbalance is a shortage ($\rho_j > 0$), then the higher the population weighted sensitivity value, the less of the item the consumer will demand. If $\hat{\gamma}_i$ or the surplus is sufficiently high or $a_{ij}$ is sufficiently low such that $a_{ij} < \hat{\gamma}_i \rho_j$, then the consumer will look to sell the quantities owned.

**Proposition 2.** *Finding a vector* $\mathbf{s}$ *which solves the LCP*

$$\mathbf{0} \leq \mathbf{s} \perp \mathbf{q} = \mathbf{r} + \mathbf{M}\mathbf{s} \geq \mathbf{0} \tag{3.6}$$

*is equivalent to solving problem (3.3)-(3.5) with correlated demand elasticities.*

**Corollary 2.** *If* $\bar{\mathbf{a}}_i > \hat{\gamma}_i \boldsymbol{\rho}$*, then the unique quantities of goods traded for each consumer group* $i$ *is defined by* $\mathbf{y}_i = \bar{\mathbf{a}}_i - \mathbf{x}_i - \hat{\gamma}_i \boldsymbol{\rho}$ *for all* $i \in \mathcal{I}$*.*

The condition that $\bar{\mathbf{a}}_i > \hat{\gamma}_i \boldsymbol{\rho}$ is equivalent to $\mathbf{r}_i > 0$. If $\mathbf{r}_i$ is positive for all consumers, then $\mathbf{s}_i = \mathbf{0}$ for all consumer groups and prices will be uniquely set such that no representative consumer completely sells out of their owned products. If $\bar{\mathbf{a}}_i > \hat{\gamma}_i \boldsymbol{\rho}$, then consumer $i$ ideal desired product set is greater than the product imbalance weighted by the consumer's normalized price sensitivity. Intuitively, if there is a product imbalance resulting in a surplus, then prices will adjust such that consumers do not sell out of a product. If there are some products where there is a shortage, then the higher relative desires for these products in conjunction with the equilibrium price prevents consumers from completely selling out of their products owned. Although the number of products traded is influenced by the initial quantities owned, the end quantities is completed determined by the input parameters. For notational convenience we denote a standard LCP of the form (3.6) by the vector matrix pair $(\mathbf{r}, \mathbf{M})$.

**Theorem 1.** *The complementarity problem (3.3)-(3.5) has a unique solution* $\mathbf{s}^*$*, which leads to a unique price vector* $\mathbf{p}^*$ *and unique trade decisions* $\mathbf{y}^*$ *in the exchange market, where* $\mathbf{p}^* = \bar{\bar{\mathbf{B}}}^{-1}\boldsymbol{\rho} + \sum_{i\in\mathcal{I}} \hat{\gamma}_i \mathbf{s}_i^*$ *and* $\mathbf{y}_i^* = \bar{\mathbf{a}}_i - \mathbf{x}_i - \bar{\mathbf{B}}_i\big(\mathbf{p}^* - \mathbf{s}_i^*\big)$ *for all* $i$*.*

Theorem 1 shows that the supply constraint multipliers $\mathbf{s}$ are independent of the state $\mathbf{x}$. This implies that the allocation of products after trade are independent of which consumers control supply. The independence of multipliers from $\mathbf{x}$ is counter-intuitive, since multipliers of capacity typically increase when capacity becomes limited. However, the presence of the fictitious agent implies that consumers have equal bargaining power. Thus, the market clearing mechanism only accounts for each representatives consumer product profile and the total supply of products. As a result, prices are also independent of the state $\mathbf{x}$. If there is product surplus, then it is feasible that $\bar{\bar{\mathbf{B}}}^{-1}\boldsymbol{\rho} \ngeq \mathbf{0}$ and prices may be negative. A sufficient condition for positive prices is $\boldsymbol{\rho} \geq 0$. In the exchange market, the aggregate surplus is the sum of each consumers realized after trade surplus and is denoted by $\bar{v} = \sum_{i\in\mathcal{I}} v_i$. A direct corollary of Theorem 1 is that the independence of $\mathbf{x}$ extends to aggregate surplus.

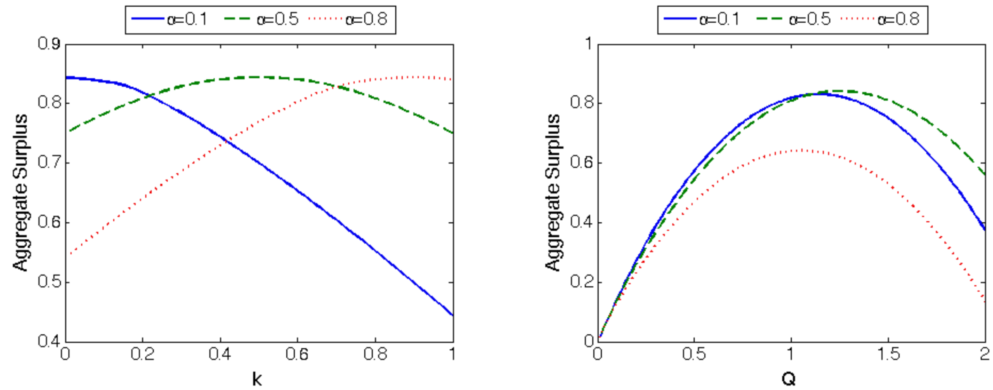
**Corollary 3.** *The aggregate surplus is independent of the initial allocation of products.*

The surplus of each consumer is the utility of owning the desired product set in addition to the cost/revenue in obtaining the quantity. Although the utility provided to each consumer is fixed, the surplus of each consumer is determined by the quantity of $\mathbf{x}_i$. Since consumer can sell undesired products, a higher quantity of $\mathbf{x}_i$ will improve the consumer $i$'s surplus. However, the relative differences in surplus caused by the initial distribution of products, due to the market clearing, sums to zero.

**Example 1.** Consider two consumers with symmetric but type specific preferences. Consumers have the same willingness to pay for their preferred and less preferred product. Specifically let $a_{ii} = 1$ and $a_{ii'} = a$ for $i \in \{1,2\}$ and $i' \neq i$. Let the population size of consumer 1 be $k$ and the population size of consumer 2 be $1-k$. Let the supply of products be described by $\bar{\bar{x}}_1 = \alpha Q$ and $\bar{\bar{x}}_2 = (1-\alpha)Q$, where $Q$ is the total number of products in the market and $\alpha$ is the proportion of total goods that are product 1. We assume that consumers have uniform sensitivities of ideal quantities due to changes in price, where the sensitivity for each group is characterized by the parameter $b \in [0,1)$. The quantity sensitivities for own prices are scaled to $-1$, so for each group $\mathbf{B} = (1+b)\mathbf{I}_n - b\mathbf{E}_n$ where $\mathbf{I}_n$ denotes a size $n$ identity matrix and $\mathbf{E}_n$ denotes an $n \times n$ matrix, all of whose elements are equal to 1.

For the scenario where $a = 0.25$, $b = 0.5$, and $Q = 1$, Figure 1 (a) shows the aggregate market surplus at various levels of $\alpha$ and $k$. Increases in $k$ increase (decrease) the aggregate market preferences for products 1 (2). As a result, the aggregate surplus for higher values of $k$ will be maximized at higher values of $\alpha$, since the distribution of products will be more aligned with consumer demand. For the scenario where $a = 0.25$, $b = 0.5$, and $k = 0.2$, Figure 1 (b) shows the aggregate market welfare at various degrees of $\alpha$ and $Q$. At $\alpha = 0.8$, the optimal quantity and maximum aggregate surplus are lower than for values of $\alpha$ where the distribution is more balanced with preferences. However, alignment between $k$ and $\alpha$ does not necessarily lead to the greater aggregate surplus. At low values of $Q$, $\alpha = 0.1$ produces greater surplus compared to $\alpha = 0.5$, since there are more products available for the larger consumer group. However, at high quantities and $\alpha = 0.1$, the availability of consumer 1's preferred product is

Figure 1: Aggregate surplus across population distribution $k$ and total quantities $Q$ for different values of $\alpha$.




(a) $Q = 1$ (b) $k = 0.2$

still below the desired level, while there is a surplus of product 2. The better balance between supply and preference results in the surplus at $\alpha = 0.5$ exceeds $\alpha = 0.1$ at higher quantities.

This example indicates that the aggregate surplus is not necessarily increasing in the number of products or the number of market participants. This observation contrasts many quantity competition results stating that consumer surplus increases (towards the level achieved under perfect competition) as the number of firms increases. Moreover, the example highlights that the aggregate consumer surplus is predicated on the balance between the consumer preferences and the number and distribution of products available in the market.

# 4. Dynamic C2C Market

The product preferences and sensitivities of consumers are dynamic over time. To understand the influence of changes in consumer preferences and the impact of strategic behavior on the quantity and price of goods traded, we extended the baseline model to a multi-period setting. At the start of each period $t \in \mathcal{T} \equiv \{1, \ldots, T\}$ in the dynamic game, the fictitious agent sets the current period prices to $\mathbf{p}_t$. Consumers have expectations on the future prices that will be set by the fictitious agent and of the goods traded in each period based on the price path and preference changes. Each consumer holds the same belief of the future prices and quantity exchanges. Considering the state parameter $\mathbf{x}_t = \begin{bmatrix}\mathbf{x}_{t1}^T \ \mathbf{x}_{t2}^T \ \ldots \ \mathbf{x}_{tm}^T\end{bmatrix}^T$, the initial quantities held across the entire market, each consumer decides $\mathbf{y}_{ti}$, the number of products to purchase and sell at market price $\mathbf{p}_t$. Right before the start of the next period $t+1$, consumers within each segment may change types. The probability that a type $i$ consumer changes consumer segments in each period is described by the vector $\boldsymbol{\lambda}_i = [\lambda_{i1} \ \lambda_{i2} \ \cdots \ \lambda_{im}]$. After consumers transition in period $t$, the next period starts and the fictitious agent sets prices to $\mathbf{p}_{t+1}$. Products are assumed to be completely durable over the course of the game, which implies that $\sum_{i \in \mathcal{I}} \mathbf{x}_{ti} = \bar{\bar{\mathbf{x}}}_t = \bar{\bar{\mathbf{x}}}$ for all $t \in \mathcal{T}$.

Since $\boldsymbol{\lambda}_i$ for all $i \in \mathcal{I}$ is an exogenous parameter known to all participants, the population size $k_{ti}$ of consumer segment $i$ in each period $t \in \mathcal{T} \equiv \{1, \ldots, T\}$ is deterministic. Although population sizes are deterministic, individual consumers make exchange decisions in the presence of uncertainty of their own future product preference and price sensitivities. The value $\|\boldsymbol{\lambda}_i\|^2$ is the average transition probability of consumer $i$ and represents the levels of uncertainty that a consumer has over their future consumer segments. Consumers have the greatest uncertainty when they are equally likely to be any type of consumer in the next period. Consumers have the greatest certainty when they transition into a specific type of consumer with probability 1. Therefore, $\frac{1}{m} \leq \|\boldsymbol{\lambda}_i\|^2 \leq 1$ for all $i \in \mathcal{I}$. To facilitate the analysis, we introduce the parameters $\omega_t = \sum_{i \in \mathcal{I}} \hat{\gamma}_{ti} \|\boldsymbol{\lambda}_i\|^2$ and $\phi_{ti} = \|\boldsymbol{\lambda}_i\|^2 - \omega_t$. The value of $\omega_t$ represents the population price sensitivity weighted certainty in period $t$. We refer to this value as the market certainty in period $t$. In the special case where $\hat{\gamma}_i$ is homogeneous across consumer groups, the market certainty is a scaled Frobenius norm and is related to the speed of convergence of consumer types. In general, the value is a weighted inner product of the transition matrix scaled by weights of each rows. The level of certainty of consumer $i$ in comparison to the market certainty is captured by the parameter $\phi_{ti}$. If $\phi_{ti} > 0$ ($\phi_{ti} < 0$) then consumer $i$ has *excess certainty* (*excess uncertainty*) of their preference evolution.

The population dynamics and objectives of the representative consumers and the fictitious agent determine the expected payoffs for the population. The fictitious agent's objective is to select a market clearing price $\mathbf{p}_t^*$, such that $\sum_{i \in \mathcal{I}} \mathbf{y}_{ti} = 0$, i.e. the quantities supplied is equalled to the quantities demanded across all products in period $t$. Representative consumer $i$'s value function is comprised of two components, the surplus earned from the exchange in the current period and the future expected value given segment dynamics and future expected exchanges and prices. The current period surplus is dependent upon the starting quantities $\mathbf{x}_t$ and the prices $\mathbf{p}_t$. The future expected value is a function of the starting quantities for the following period $\mathbf{x}_{t+1}$. We assume that the fictitious agent and consumers all have rational expectations. They correctly anticipate the behavior of the consumers and can compute their future expected payoffs. We initially assume that the next period market clearing prices $\mathbf{p}_{t+1}^*(\mathbf{x}_{t+1})$ have dependence upon state $\mathbf{x}_t$. The future expected value for a type $j$ consumer is $v_{(t+1)j}^*(\mathbf{x}_{t+1}) = v_{(t+1),j}(\mathbf{x}_{t+1}, \mathbf{p}_{t+1}^*(\mathbf{x}_{t+1}))$. Thus, for a given set of price $\mathbf{p}_t$, the value function of consumer $i$ is

$$v_{ti}(\mathbf{x}_t, \mathbf{p}_t) = \max_{\mathbf{y}_{ti}} \quad \big(\mathbf{x}_{ti} + \mathbf{y}_{ti}\big)^T \left(\tilde{\mathbf{a}}_i - \frac{1}{2k_{ti}} \tilde{\mathbf{B}}_i \big(\mathbf{x}_{ti} + \mathbf{y}_{ti}\big)\right) - \mathbf{y}_{ti}^T \mathbf{p}_t + \delta \sum_{j \in \mathcal{I}} \lambda_{ij} v_{(t+1)j}^*(\mathbf{x}_{t+1}) \tag{4.1}$$

$$\text{s.t.} \qquad \mathbf{x}_{ti} + \mathbf{y}_{ti} \geq \mathbf{0}, \tag{4.2}$$

$$\mathbf{x}_{(t+1)i} = \sum_{j \in \mathcal{I}} \lambda_{ji} \left(\mathbf{x}_{tj} + \mathbf{y}_{tj}\right), \quad \forall i \in \mathcal{I} \tag{4.3}$$

with the termination condition $\mathbf{v}_{(T+1)i}(\mathbf{x}_{T+1}) = 0, \quad \forall i \in \mathcal{I}, \quad \mathbf{x}_{T+1} \in \Re^n$. The parameter $\delta \in [0, 1]$ represents the level of strategic behavior. Thus, the exchange market is a stochastic game, where each representative consumer $i$ solves (4.1)-(4.3) and the fictitious agent selects $\mathbf{p}_t^*$ such that

$$\sum_{i \in \mathcal{I}} \mathbf{y}_{ti} = \mathbf{0}, \quad \forall t \in \mathcal{T}. \tag{4.4}$$

Observe that if $\delta = 0$, then consumers are myopic and make decisions entirely based on the current periods utility. As a result when $\delta = 0$, the multi-period game reduces to $T$ static $C2C$

games, where the populations and starting period product distributions evolve according to the transition dynamics. We now present the main theorem of the dynamic C2C market.

**Theorem 2.** *The equilibrium in the multi-period exchange market game has the following properties for each $t \in \mathcal{T}$:*

1. *The supply multiplier values $\mathbf{s}_t^*$ are unique in each period $t$ and independent of $\mathbf{x}_t$.*
2. *The price $\mathbf{p}_t^*$ is unique, independent of $\mathbf{x}_t$. and equals*

$$\mathbf{p}_t^* = \bar{\bar{\mathbf{B}}}_t^{-1}\boldsymbol{\rho}_t + \sum_{i \in \mathcal{I}} \hat{\gamma}_{ti}\mathbf{s}_{ti}^* + \delta\omega_t\mathbf{p}_{t+1}^* \tag{4.5}$$

3. *The quantity of products exchanged in each period by consumer $i$ is unique and equals*

$$\mathbf{y}_{ti}^* = \bar{\mathbf{a}}_{ti} - \mathbf{x}_{ti} - \hat{\gamma}_{ti}\boldsymbol{\rho}_t + \delta\phi_{ti}\bar{\mathbf{B}}_{ti}\mathbf{p}_{t+1}^* + \bar{\mathbf{B}}_{ti}\left(\mathbf{s}_{ti}^* - \sum_{i' \in \mathcal{I}} \hat{\gamma}_{ti'}\mathbf{s}_{ti'}^*\right). \tag{4.6}$$

4. *The consumer optimization problem (4.1)-(4.3) is equivalent to solving*

$$\begin{aligned} v_{ti}(\mathbf{x}_t, \mathbf{p}_t) = \max_{\mathbf{y}_{ti} \geq -\mathbf{x}_{ti}} \quad & \big(\mathbf{x}_{ti} + \mathbf{y}_{ti}\big)^T \left(\tilde{\mathbf{a}}_i - \frac{1}{2k_{ti}}\tilde{\mathbf{B}}_i\big(\mathbf{x}_{ti} + \mathbf{y}_{ti}\big)\right) - \mathbf{y}_{ti}^T\mathbf{p}_t \\ & + \delta \sum_{j \in \mathcal{I}} \lambda_{ij}\left(c_{(t+1)j} + \sum_{j' \in \mathcal{I}} \lambda_{j'j}\big(\mathbf{x}_{tj'} + \mathbf{y}_{tj'}\big)^T \mathbf{p}_{t+1}^*\right), \end{aligned} \tag{4.7}$$

   *where $c_{(t+1)j}$ and $\mathbf{p}_{t+1}^*$ are constants entirely determined by input parameters.*
5. *The aggregate surplus is independent of the initial allocation of products in all periods.*

Similar to the static game, Theorem 2 shows that the supply constraint multipliers in each period $\mathbf{s}_t^*$ are independent of the state $\mathbf{x}_t$. Although the model set allowed for prices to depend on the state $\mathbf{x}_t$, the theorem demonstrates that in equilibrium, prices remain independent of the state. Rather, prices are dependent upon market dynamics and the level of strategic behavior. Although the quantity of products traded in any period is conditional on the starting quantity vector, adding $\mathbf{x}_{ti}$ to both sides of (4.6) shows that the end quantities owned, $\mathbf{q}_{ti} = \mathbf{x}_{ti} + \mathbf{y}_{ti}$, is independent of the state $\mathbf{x}_{ti}$. The market clearing mechanism effectively sets prices to allocate products amongst consumers to maximize utility based on preferences profiles and supply, which is independent of the state.

Following the approach for the static model, to prove Theorem 2, we converted the multi-period problem into a standard LCP. For $t \in \mathcal{T}$ define the vectors $\mathbf{q}_t$ and $\mathbf{r}_t$ as

$$\mathbf{q}_t = \begin{bmatrix} \mathbf{x}_{t1} + \mathbf{y}_{t1} \\ \mathbf{x}_{t2} + \mathbf{y}_{t2} \\ \vdots \\ \mathbf{x}_{tm} + \mathbf{y}_{tm} \end{bmatrix} \qquad \text{and} \qquad \mathbf{r}_t = \begin{bmatrix} \bar{\mathbf{a}}_{t1} - \hat{\gamma}_{t1}\boldsymbol{\rho}_t + \delta\phi_{t1}\bar{\mathbf{B}}_{t1}\mathbf{p}_{t+1}^* \\ \bar{\mathbf{a}}_{t2} - \hat{\gamma}_{t2}\boldsymbol{\rho}_t + \delta\phi_{t2}\bar{\mathbf{B}}_{t2}\mathbf{p}_{t+1}^* \\ \vdots \\ \bar{\mathbf{a}}_{tm} - \hat{\gamma}_{tm}\boldsymbol{\rho}_t + \delta\phi_{tm}\bar{\mathbf{B}}_{tm}\mathbf{p}_{t+1}^* \end{bmatrix},$$

and the block matrix $\mathbf{M}_t$ as

$$\mathbf{M}_t = \begin{bmatrix} \bar{\gamma}_{t1}(1-\hat{\gamma}_{t1})\mathbf{B} & -\bar{\gamma}_{t1}\hat{\gamma}_{t2}\mathbf{B} & \cdots & -\bar{\gamma}_{t1}\hat{\gamma}_{tm}\mathbf{B} \\ -\bar{\gamma}_{t2}\hat{\gamma}_{t1}\mathbf{B} & \bar{\gamma}_{t2}(1-\hat{\gamma}_{t2})\mathbf{B} & \cdots & -\bar{\gamma}_{t2}\hat{\gamma}_{tm}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\gamma}_{tm}\hat{\gamma}_{t1}\mathbf{B} & -\bar{\gamma}_{tm}\hat{\gamma}_{t2}\mathbf{B} & \cdots & \bar{\gamma}_{tm}(1-\hat{\gamma}_{tm})\mathbf{B} \end{bmatrix}.$$

The proof shows that the unique optimal price path and the equilibrium quantities traded can be found by recursively solving the LCP $(\mathbf{r}_t, \mathbf{M}_t)$. Since the market parameters are known, the quantities following trades can be calculated if the future price is known. Starting with $\mathbf{p}^*_{T+1} = \mathbf{0}$, we can solve LCP $(\mathbf{r}_T, \mathbf{M}_T)$ for the unique multipliers $\mathbf{s}^*_T$ to find prices $\mathbf{p}^*_T$. Given the value of $\mathbf{p}^*_T$ we can solve $(\mathbf{r}_{T-1}, \mathbf{M}_{T-1})$ for multipliers $\mathbf{s}^*_{T-1}$ in order to find prices $\mathbf{p}^*_{T-1}$. Thus, by recursively solving LCPs $(\mathbf{r}_\tau, \mathbf{M}_\tau)$ for $\tau \in \{T, T-1, \cdots, t\}$, we can find the values of $\mathbf{s}^*_t$, providing prices $\mathbf{p}^*_t$. Since the end allocations of products and prices are determined by recursively solving the LCP values and do not depend on $\mathbf{x}_t$, similar to the static game, the aggregate surplus in any period $t$ is also independent of the product distribution.

Each consumer's value function can be separated into the current utility from owning a collection of products, the cost from buying items (or revenue from selling), and the expected future value. Substituting the expression for $\mathbf{q}^*_{ti}$ into (4.7) reveals that the current utility and future value are independent of the state. Thus, only the cost component of the value function is (linearly) dependent upon $\mathbf{x}_t$. In addition, the transition dynamics are independent of consumer decisions due to the surplus maximizing market clearing mechanism. These factors lead to a Markov perfect equilibrium, given the solutions to the associated LCPs. Similar separability approaches have been used in other recent studies of competitive dynamic games (for example Liu et al. (2007), Olsen and Parker (2008), Nagarajan and Rajagopalan (2009), Olsen and Parker (2014); and Yang and Zhang (2015)). The independence of the aggregate surplus results from the fact that the cost component of the value function sums to zero after aggregating the value functions.

## 4.1 Impact of Strategic Behavior on the Equilibrium Solution

In this section we explore the impact of strategic behavior on the equilibrium prices, after trade quantities, and aggregate surplus. The prices in each period $t$ are comprised of a current value $\mathbf{p}_{tc}$ and a future value $\mathbf{p}_{tf}$ component, such that $\mathbf{p}^*_t = \mathbf{p}^*_{tc} + \mathbf{p}^*_{tf}$. From the price equation (4.5) in Theorem 2, $\mathbf{p}^*_{tc} = \bar{\bar{\mathbf{B}}}_t^{-1}\boldsymbol{\rho}_t + \sum_{i\in\mathcal{I}} \hat{\gamma}_{ti}\mathbf{s}^*_{ti}$ and $\mathbf{p}^*_{tf} = \delta\omega_t \mathbf{p}^*_{t+1}$. The current period component $\mathbf{p}^*_{tc}$ is based on the value of products given the current population segment sizes and active supply constraints. Since the $\mathbf{p}^*_{tc}$ is unconcerned with the future value of the product, it has the same functional form as the static game prices. The future component is the next period price multiplied by the scalar $\delta\omega_t$. The term $\delta\omega_t$ is the interaction between the market level of strategic behavior and the market level of uncertainty. This interaction term is between 0 and 1, and its weight determines the extent to which the future value of the product is incorporated into the current period price. The greater the average transition probability the more certain consumers are of their future product preferences. Thus, the greater the levels of certainty, price sensitivity, and stategicity, the greater the influence of the next period prices on the current prices.

The insights into strategic behavior and preference changes are particularly relevant for the collectibles market. The structural property that market certainty positively influences price is supported by the empirical evidence for fine art. Although contemporary art is subjected to fads, preferences for traditional art movements such as Romanticism and Baroque are mostly constant amongst collectors (Penasse and Renneboog 2014). As a result, consumer dynamics for masterpieces pertain to changes in collectors' wealth rather than changes in preferences for artists (Parker and Vissing-Jorgensen 2010; Goetzmann et al. 2011). In a recent study on prices and returns in the art market, Renneboog and Spaenjers (2013) find convincing evidence that the price of fine art is positively affected by high-income consumers' confidence over their future income expectations.

**Corollary 4.** *There exists a threshold* $\tilde{\delta}^p_j = \frac{1}{\omega_t}\left(1 - \frac{\left(\bar{\bar{\mathbf{B}}}_t^{-1}\boldsymbol{\rho}_t\right)_j}{p^*_{(t+1)j}}\right)$*, such that the price for product* $j$ *is nonincreasing for all* $\delta \geq \tilde{\delta}^p_{tj}$ *from period* $t$ *to period* $t+1$*. Furthermore, if no consumer faces supply constraints for product* $j$ *in period* $t$*, then the price for product* $j$ *will always increase from period* $t$ *to period* $t+1$ *if* $\delta < \tilde{\delta}^p_{tj}$*.*

The threshold result demonstrates that strategic behavior may not influence whether price increases or decreases for certain inputs. For example, if $p^*_{tcj} > p^*_{(t+1)j} > 0$, where $\mathbf{s}^*_{ti} = 0$ for all $i \in \mathcal{I}$, then the threshold value $\tilde{\delta}^p_{tj}$ will be negative, implying prices will be nonincreasing between periods for all levels of strategic behavior. If $(1-\omega_t)p^*_{(t+1)j} \geq p^*_{tcj} \geq 0$, then $\tilde{\delta}^p_{tj}$ will exceed 1 and the price of $j$ will increase irrespective of the level of strategic behavior. However, for moderate dynamics such as $0 < p^*_{tcj} < p^*_{(t+1)j}$ where $p^*_{tcj} > (1-\omega_t)p^*_{(t+1)j}$, the level of strategic behavior will dictate the direction of price.

From equation (4.6), the end quantities of products owned by consumer $i$ in period $t$ are $\mathbf{q}^*_{ti} = \bar{\mathbf{a}}_{ti} - \hat{\gamma}_{ti}\boldsymbol{\rho}_t + \delta\phi_{ti}\bar{\mathbf{B}}_{ti}\mathbf{p}^*_{t+1} + \bar{\mathbf{B}}_{ti}\left(\mathbf{s}^*_{ti} - \sum_{i'\in\mathcal{I}}\hat{\gamma}_{ti'}\mathbf{s}^*_{ti'}\right)$. Although $\delta$ only explicitly enters into the term $\delta\phi_{ti}\bar{\mathbf{B}}_{ti}\mathbf{p}^*_{t+1}$, the parameter also implicitly impacts supply multipliers and future prices. The consumer price sensitivity $\gamma_i$ enters into each of the individual parameters in the term $\phi_{ti}\bar{\mathbf{B}}_{ti}\mathbf{p}^*_{t+1}$, as well as the terms $\hat{\gamma}_{ti}\boldsymbol{\rho}_t$ and $\bar{\mathbf{B}}_{ti}\left(\mathbf{s}^*_{ti} - \sum_{i'\in\mathcal{I}}\hat{\gamma}_{ti'}\mathbf{s}^*_{ti'}\right)$. Similarly, with respect to price, the expression (4.5) is nonlinear in $\delta$ in a multi-period setting.

The complex dependency of end period quantities and prices on $\delta$ deserve consideration in a simpler setting in order to obtain a more complete characterization of the properties. in a simpler setting in order to obtain a more complete characterization of the properties. Subsequently, the following results and examples consider the setting of $T = 2$ and/or $\mathbf{s}^*_{ti} = \mathbf{0}$. For $T = 2$, the final period prices do not depend on the value of $\delta$. When the solution of a multi-period game has $\mathbf{s}_{ti} = 0$ in period $t$, then the equilibrium is characterized by *comprehensive ownership* of products in period $t$. Each consumer owning some quantity of each product offers the simplicity that with knowledge of the price vector $\mathbf{p}_{t+1}$, the equilibrium quantities and prices can be expressed without solving a LCP.

**Proposition 3.** *If the inequality*

$$\tilde{a}_{tij} - \left(\bar{\bar{\mathbf{B}}}_t^{-1}\boldsymbol{\rho}_t\right)_j + \delta\phi_{ti}p^*_{(t+1)j} \geq 0 \quad \forall i \in \mathcal{I}, j \in \mathcal{J}, \tag{4.8}$$

*is satisfied in period* $t$*, then the equilibrium solution has* $\mathbf{s}^*_{ti} = \mathbf{0}$*.*

Proposition 3 characterizes the set of market parameters where each consumer in equilibrium owns quantities of each product. If condition (4.8) is satisfied for all $\delta \in [0,1]$, then the equilibrium in period $t$ has comprehensive ownership for all levels of strategic behavior. Depending on the input parameters, there is a range of levels of strategic behavior where local changes in $\delta$ do not impact the comprehensive ownership of products. The following corollary identifies threshold values of strategicity where the solution exhibits the property of comprehensive ownership in period 1 for a two period game.

**Corollary 5.** *Consider $T = 2$. If consumers are not homogeneous in terms of their level of strategic certainty $\delta\|\lambda_i\|^2$ and there exists at least one non-zero price $p_{2j}$, then comprehensive ownership occurs for $\tilde{\delta}^{s-} < \delta < \tilde{\delta}^{s+}$, where*

$$\tilde{\delta}^{s+} = 1 \wedge \min\left\{ \frac{\tilde{a}_{1ij} - \left(\bar{\bar{\mathbf{B}}}_1^{-1}\boldsymbol{\rho}_1\right)_j}{-\phi_{1i}\mathbf{p}_{2j}^*} \middle| i \in \mathcal{I}, j \in \mathcal{I}, -\phi_{1i}\mathbf{p}_{2j}^* > 0 \right\} \qquad \textit{and}$$

$$\tilde{\delta}^{s-} = 0 \vee \max\left\{ \frac{\tilde{a}_{1ij} - \left(\bar{\bar{\mathbf{B}}}_1^{-1}\boldsymbol{\rho}_1\right)_j}{-\phi_{1i}\mathbf{p}_{2j}^*} \middle| i \in \mathcal{I}, j \in \mathcal{J}, -\phi_{1i}\mathbf{p}_{2j}^* < 0 \right\}.$$

*If consumers are homogeneous in terms of their level of certainty or $\mathbf{p} = \mathbf{0}$ then comprehensive ownership occurs for all $\delta \in [0, 1]$ if and only if $\tilde{a}_{tij} - \left(\bar{\bar{\mathbf{B}}}_t^{-1}\boldsymbol{\rho}_t\right)_j \geq 0 \quad \forall i \in \mathcal{I}, j \in \mathcal{I}$.*

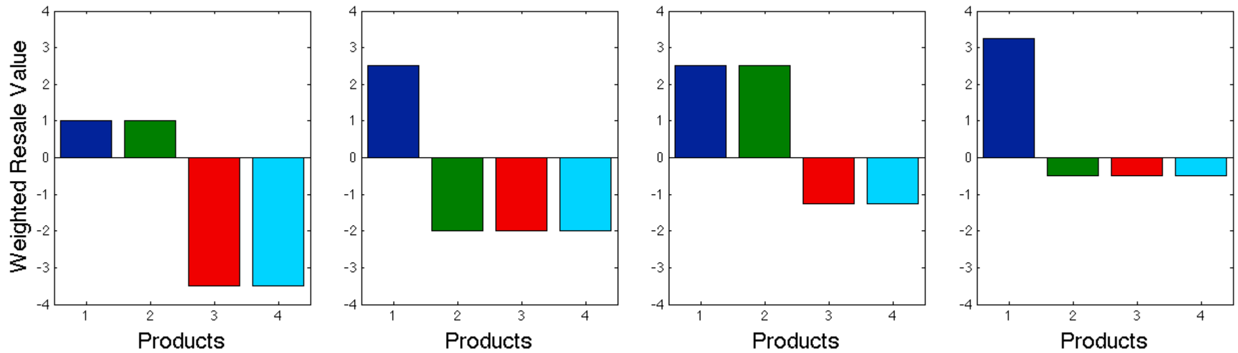
If $\tilde{\delta}^{s-} = 0$ and $\tilde{\delta}^{s+} = 1$, then changes in the market level of strategic behavior will not force a consumer group to sell out of any product. If $\tilde{\delta}^{s-} \neq 0$ or $\tilde{\delta}^{s+} \neq 1$, then the thersholds in Corollary 5 can identify the consumer group that will be the first to sell out of a product given a change in $\delta$. For example, when $\mathbf{p}_2^* > 0$ and $\tilde{\mathbf{a}}_{1i} - \bar{\bar{\mathbf{B}}}_1^{-1}\boldsymbol{\rho}_1 \geq \mathbf{0}$ holds for each $i \in \mathcal{I}$, only consumers with excess uncertainty will sell their entire stock of a product as consumers become more strategic. A consumer who has a sufficiently low willingness to pay for a product that is highly desired will stock out if the expected future price of the product and/or the consumer's relative uncertainty is high.

For $T = 2$, $\tilde{\delta}^{s-} = 0$, and $\tilde{\delta}^{s+} = 1$, the quantities owned by consumers at the end of period 1 are influenced by the interaction between preference certainty and the price sensitivity weighted resale value. The resale value (second period prices) reflects the future product imbalance (ideal ownership compared to the total supply) and substitutability of products (market price sensitivities). When consumers are strategic, the interaction effect between certainty and resale value results in consumer $i$ owning either more or less of product $j$ compared to the case of myopic consumers. Both the weighted resale value and preference certainty (compared to the market level) can be either positive or negative. This creates four cases of how strategic behavior impacts the period 1 quantities owned. The four cases are summarized in Table 1 and state that high (low) resale value and excess certainty (uncertainty) result in a strategic consumer owning more of a product than in the case where consumers are myopic. The magnitude of the change in quantity is increasing in the magnitude of the resale value and preference certainty and is increasing in the level of strategic behavior.

Table 1: Comparison between consumer $i$'s equilibrium quantities for product $j$ at $\delta > 0$ and at $\delta = 0$ (indicated by superscripts) given the weighted resale value of $j$ and preference certainty of $i$.

| | Excess Certainty $\phi_i > 0$ | Excess Uncertainty $\phi_i < 0$ |
|---|---|---|
| Positive Weighted Resale Value $(\bar{\mathbf{B}}_{1i}\mathbf{p}_2^*)_j > 0$ | $q_{1ij}^{\delta>0} > q_{1ij}^{\delta=0}$ | $q_{1ij}^{\delta>0} < q_{1ij}^{\delta=0}$ |
| Negative Weighted Resale Value $(\bar{\mathbf{B}}_{1i}\mathbf{p}_2^*)_j < 0$ | $q_{1ij}^{\delta>0} < q_{1ij}^{\delta=0}$ | $q_{1ij}^{\delta>0} > q_{1ij}^{\delta=0}$ |

Figure 2: Weighted resale values for the set of products at low and high values of $p_2$ and $b$.




(a) ($p_2 = 4, b = 0.5$) (b) ($p_2 = 1, b = 0.5$) (c) ($p_2 = 4, b = 0.25$) (d) ($p_2 = 1, b = 0.25$)

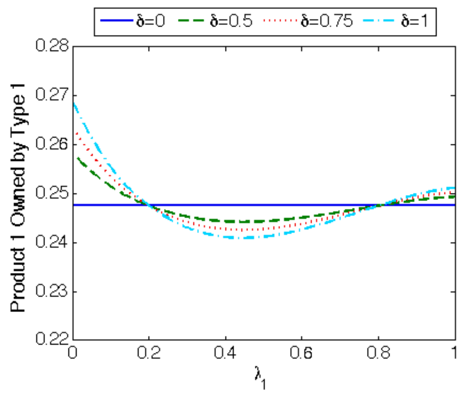
**Example 2.** Consider a two period market with four products where the prices for the second period are given by the vector $\mathbf{p}_2 = [4, p_2, 1, 1]$, and consumers have quantity sensitivity to prices given by $\mathbf{B} = (1 + b)\mathbf{I}_4 - b\mathbf{E}_4$. Figure 2 denotes the weighted resale values for $p_2 \in \{1, 4\}$ and $b \in \{0.25, 0.5\}$. When $p_2$ is high, the weighted resale value of products 3 and 4 is more negative than in the case where $p_2$ is low. However, when $p_2$ is low, product 1 becomes relatively more valuable, which increases its weighted resale value. The figure also shows that a lower $b$ increases the weighted resale value across all products, since there is less substitutability between products.

If a consumer is strategic with excess certainty and $p_2$ is low, then the consumer will increase the quantities held of product 1 compared to the case where the consumer is myopic because of its high second period relative value. On the other hand, a consumer with excess uncertainty will increase the quantities held of products 2, 3, and 4. If $p_2$ is high, then the consumer with excess certainty will be less interested in product 1 compared to the case when $p_2$ is low, but will still increase the quantities held of products 1 and 2 because of the two products positive resale value. A consumer with excess uncertainty will reduce the quantities held of products 1 and 2, and increase the quantities held of products 3 and 4 beyond the case of low $p_2$, due to the decrease in the weighted resale value.

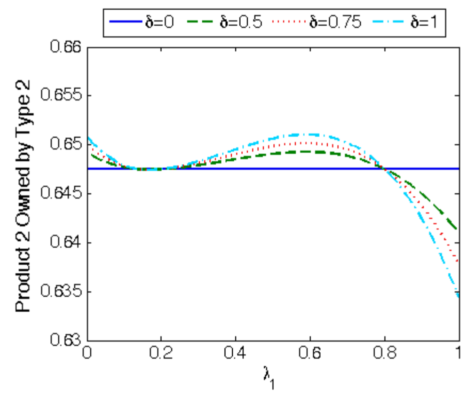
The examples with $p_2 = 1$ parallels collectible markets, since collectibles often consist of a greater proportion of lower valued items relative to the number of high value items. Novice collectors have greater uncertainty over their future preferences compared to seasoned collectors, which leads to excess uncertainty. The example suggests that novice collectors will buy a variety of low value product. On the other hand, experienced collectors sell more lower valued products to the novice collectors, enabling them to increase the quantities held of high value products, improving their future financial position. To an extent, this behavior is analogous to speculation and hedging in financial markets. The following example further illustrates the interactions between the population dynamics and strategic behavior.

**Example 3.** Consider the type specific framework from Example 1. To extend the setup to a two period setting, the transition vectors for consumer 1 and 2 are $\boldsymbol{\lambda}_1 = [1 - \lambda_1, \lambda_1]$ and $\boldsymbol{\lambda}_2 = [\lambda_2, 1 - \lambda_2]$, respectively. For an initial population vectors of $\mathbf{k}_1 = [k, 1 - k]$, the period 2 population is $\mathbf{k}_2 = [(1-\lambda_1)k+\lambda_2(1-k), \lambda_1 k+(1-\lambda_2)(1-k)]$. We set $k = \alpha = 0.3$, $\gamma_1 = \gamma_2 = 1$,

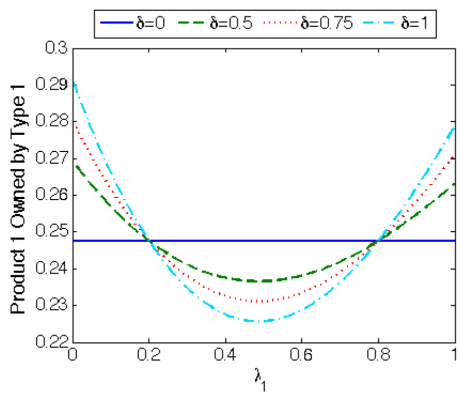
Figure 3: First period quantities after trade of preferred product for different $\delta$ across $\lambda_1$.



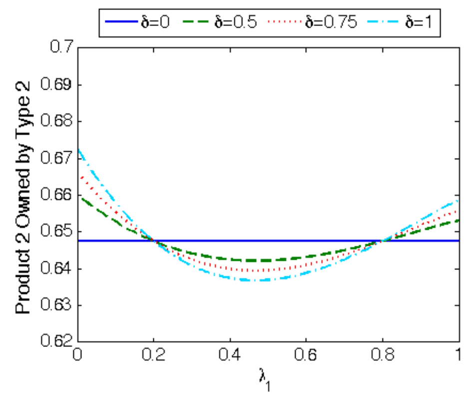

(a) $q_{111}$ with $\lambda_2 = 0.2$




(b) $q_{122}$ with $\lambda_2 = 0.2$




(c) $q_{111}$ with $\lambda_2 = 0.8$




(d) $q_{122}$ with $\lambda_2 = 0.8$

and use the previous values of $a = 0.25$, $b = 0.5$, and $Q = 1$. Since $k = \alpha = 0.3$, the proportion of products is aligned with the population in the first period. For $t = 1$, Figure 3 shows the end-period quantities of preferred product for each consumer across $\lambda_1$ for $\lambda_2 \in \{0.2, 0.8\}$ and $\delta \in \{0, 0.5, 0.75, 1\}$. Observe from Figure 3 that $0 < q_{11} < 0.3 = \bar{\bar{x}}_1$ and $0 < q_{12} < 0.7 = \bar{\bar{x}}_2$ for all $\delta$ and various values of $\lambda_1$ and $\lambda_2$. This implies that the inequality (4.8) is satisfied for all $i \in \mathcal{I}$, $j \in \mathcal{J}$, and $\delta \in [0, 1]$, and that the quantities in Example 3 has comprehensive ownership of products in period 1 for all levels of strategic behavior.

The region of excess certainty (excess uncertainty) for consumer 1 is $0 \leq \lambda_1 < 0.2$ and $0.8 < \lambda_1 \leq 1$ ($0.2 < \lambda_1 < 0.8$). Since there are only two types of consumers in the market, when consumer 1 has excess certainty, consumer 2 has excess uncertainty, and vice versa. For regions where consumer 1 has excess certainty, higher $\delta$ leads to greater quantities owned of product 1. As the fraction $\lambda_2$ of type 2 consumers shifting to type 1 increases, the period 2 ideal quantities for product 1 and hence price increases. As a result, the value of $q_{111}$ at a given level of $\delta$ increases with $\lambda_2$. This can be seen by comparing the regions of excess certainty for consumer 1 in Figures 3 (a) and (c).

If the ideal quantity for product 1 in period 2 is sufficiently high, then the weighted resale value of product 2 becomes negative, making product 2 undersirable. In Figure 3 (b), product

2 is undesirable for low values of $\lambda_1$ where consumer 2 has excess uncertainty. The negative weighted resale values and consumer 1's excess certainty, makes consumer 2 purchases more of her/his preferred product. For $\lambda_1 > 0.2$, product 2 has a positive weighted resale value and so consumer 2 purchases more of its preferred product when it has excess certainty. In Figure 3 (d) where $\lambda_2 = 0.8$, the weighted resale value of product 2 is negative for all $\lambda_1 \in [0, 1]$. Thus, excess uncertainty for consumer 2 causes greater ownership of product 2, even though the consumer is unlikely to want the product in the next period. In all four cases, at $\lambda_1 = 0.2$ or $\lambda_1 = 0.8$ neither consumer has excess certainty, and the end quantities for all products are independent of strategic behavior. When $\lambda_1 \neq 0.2$ and $\lambda_1 \neq 0.8$, the deviation from the quantities owned when consumers are myopic is increasing in the level of strategic behavior.

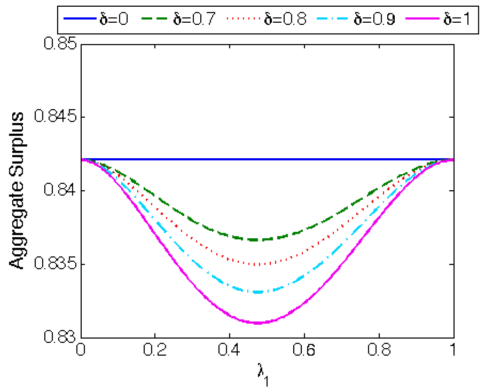
To study the influence of strategic behavior on aggregate surplus, we analyze the special cases of comprehensive ownership in a two period setting. The consumers surplus in each period consists of a quadratic utility component and value component based on the market clearing prices and trade decisions. Due to market clearing, the aggregate surplus in period $t$ is the sums of the utilities of each consumer and equals $\bar{\bar{v}}_t = \sum_{i \in \mathcal{I}} \mathbf{q}_{ti}^{*T} \big( \tilde{\mathbf{a}}_i - \frac{1}{2k_{ti}} \tilde{\mathbf{B}}_i \mathbf{q}_{ti}^* \big)$. The aggregate realized surplus is then $\bar{\bar{v}} = \sum_{t \in \mathcal{T}} \bar{\bar{v}}_t$.

**Proposition 4.** *For $T = 2$ and input parameters such that $\tilde{\delta}^{s-} = 0$ and $\tilde{\delta}^{s+} = 1$, realized aggregate realized surplus is concave in the strategic behavior. If $\phi_{1i}$ for all $i \in \mathcal{I}$ are not identically* $0$*, the level of strategic behavior on the interval* $[0, 1]$ *that maximizes aggregate surplus is* $0$*. If $\phi_{1i} = 0$ for all $i \in \mathcal{I}$, then strategic behavior does not influence aggregate surplus.*

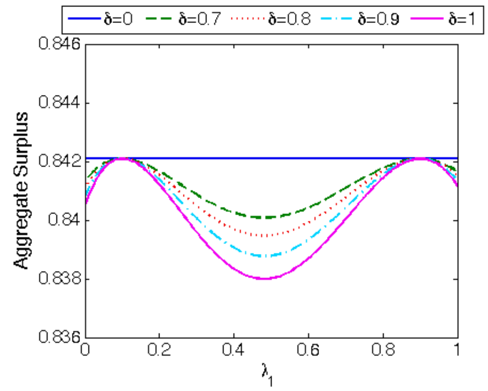
The condition that $\phi_{1i} = 0$ for all $i \in \mathcal{I}$, only occurs when each representative consumer has the same average transition probability $\|\lambda_i\|^2$. Thus, unless consumers are homogeneous in terms of their level of uncertainty, Proposition 4 shows that aggregate realized surplus is maximized when consumers are myopic. Strategic behavior causes consumers with greater certainty relative to the market average to engage in prospective behavior for products with high resale value. Consumers either want to purchase the product due to the resale value, or want to purchase the product in anticipation of a price increase of a likely future desired product. This *resale effect* causes price increases for products with high future values, irrespective of the utility structure and product preferences of consumers in the current period. Consequently, consumers who have high utility for products with high resale value and excess uncertainty in period one will deviate from purchasing preferred products due to higher prices. In addition, the resale effect implies that consumers with excess certainty are purchasing products based on value rather than utility. Since the value consumers receive from resale does not benefit the aggregate surplus, the market is most efficient when consumers are myopic.

**Example 4.** Consider the type specific framework from Example 3 with $a = 0.25$, $b = 0.5$, $Q = \gamma_1 = \gamma_2 = 1$, and $k = \alpha = 0.3$. Again in this example, the distribution of products is aligned with the first period population. Figure 4 plots the aggregate first period surplus for different levels of $\delta$ and $\lambda_2$ across $\lambda_1$. We exclude the second period surplus from the figures because it is unaffected by $\delta$. The figures clearly illustrates that aggregate surplus is maximized at $\delta = 0$ and that surplus is non-increasing in strategicity. Furthermore, as uncertainty levels become more homogeneous (decreases in $\lambda_2$), even when consumers are completely strategic, the market approaches the aggregate surplus of a myopic market. This occurs because markets

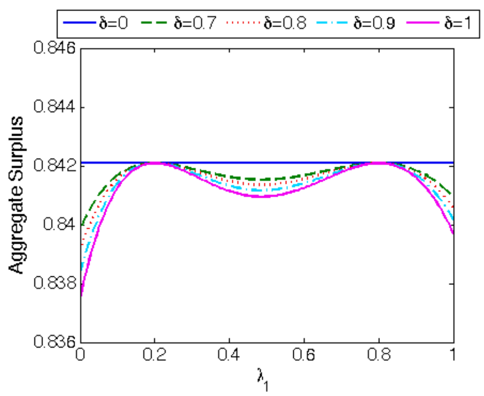
Figure 4: First period aggregate surplus across $\lambda_1$ for different values of $\lambda_2$ and $\delta$.



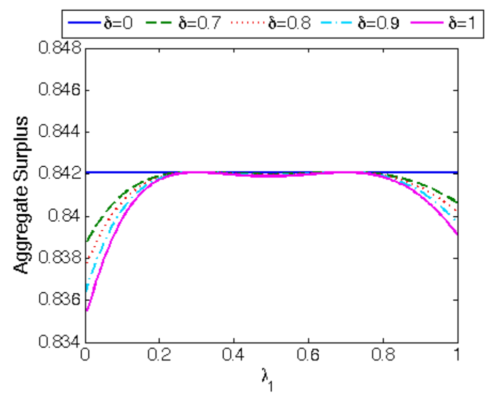

(a) $\lambda_2 = 1$




(b) $\lambda_2 = 0.9$




(c) $\lambda_2 = 0.8$




(d) $\lambda_2 = 0.7$

where consumers have greater uncertainty are less sensitive to the impact of strategic behavior. Thus, greater consumer uncertainty is beneficial for maximizing the aggregate surplus of an exchange market, since it counteracts consumer strategicity.

An influx of novice participants into a collectible market can result in greater heterogeneity in certainty levels between consumers. The example shows that the market surplus reaches a minimum when the two types of consumers have the greatest discrepancy in certainty levels. This result helps to illustrate why C2C markets become less efficient when the number of participants increases (Boone and Pottersz 2006; Kauffman et al. 2009). Previous studies have attributed this fact to a lack of information on the true value of a collectible. However, we show that a similar effect occurs, even when participants can accurately forecast the value of products.

# 5. Stationary Infinite Horizon C2C Market

If the time horizon for the game is sufficiently long or infinite, then the population segment size may reach a steady state. If the system is in a steady state at time $\tau$ with respect to population segment transitions, then $k_{ti} = k_{(t+1)i}$ for all $i \in \mathcal{I}$ and $t \in \{\tau, \ldots, T\}$. When the market is in a steady state, the market parameters remain constant, even though consumers continue to transition between segments and have uncertainty in their future preference profile. In the infinite horizon setting the nonnegative strategicity parameter $\delta$ is restricted to be strictly less than one. Before studying surplus in a game without active multipliers, it is necessary to establish if there exists a stationary solution in an infinite horizon problem with $\mathbf{s}_i = \mathbf{0}$. For notational convenience, we remove the time index for all variables and input parameters when the system is in steady state.

**Proposition 5.** *If the system is in a steady state in an infinite horizon game and*

$$\bar{a}_{ij} - \hat{\gamma}_i \rho_j - \delta\big(\omega \bar{a}_{ij} - \|\boldsymbol{\lambda}_i\|^2 \hat{\gamma}_i \rho_j\big) \geq 0 \quad \forall i \in \mathcal{I}, j \in \mathcal{J}, \tag{5.1}$$

*then there exists an equilibrium solution where* $\mathbf{s}_i^* = \mathbf{0}$, $\mathbf{p}^* = \frac{1}{1-\delta\omega}\bar{\bar{\mathbf{B}}}^{-1}\boldsymbol{\rho}$ *and* $\mathbf{q}_i^* = \bar{\mathbf{a}}_i - \hat{\gamma}_i \boldsymbol{\rho} + \frac{\delta \phi_i \hat{\gamma}_i}{1-\delta\omega}\boldsymbol{\rho}$.

The proposition shows a sufficient condition for the existence of the stationary solution with $\mathbf{s}_i^* = \mathbf{0}$ for all $i \in \mathcal{I}$. Condition (5.1) provides the range of levels of strategic behavior for which a stationary solution is guaranteed to exist. We refer to the set of stationary solutions such that (5.1) is satisfied as comprehensive stationary markets. In comprehensive stationary markets, the magnitude of the steady state price is increasing in the level of strategic behavior and market certainty. For product $j$, if $\big(\bar{\bar{\mathbf{B}}}^{-1}\boldsymbol{\rho}\big)_j \geq 0$ $\big(\big(\bar{\bar{\mathbf{B}}}^{-1}\boldsymbol{\rho}\big)_j \leq 0\big)$, then strategic behavior and consumer certainty will increase (decrease) the price for solutions where (5.1) holds.

In a stationary setting, the product imbalance $\boldsymbol{\rho}$ and the market weighted sensitivities $\bar{\bar{\mathbf{B}}}$ are identical in each period. As a result, the lifetime consumption value of each product in an infinite horizon setting is static. Despite this fact, the stationary prices for products are still dependent upon market dynamics, where higher levels of market certainty lead to higher prices. This provides evidence that the prices for products such as collectibles are dependent upon factors outside the consumption value of items, even in a setting where all market participants are entirely rational and transactions are frictionless.

**Corollary 6.** *If there exists* $i \in \mathcal{I}$ *and* $j \in \mathcal{J}$ *such that* $\bar{a}_{ij} - \hat{\gamma}_i \rho_j < 0$ *and* $\omega \bar{a}_{ij} - \|\boldsymbol{\lambda}_i\|^2 \hat{\gamma}_i \rho_j \geq 0$*, then condition (5.1) will not hold for any* $\delta \in [0, 1)$*, otherwise condition (5.1) holds for* $\max\{\tilde{\delta}^{s-}, 0\} \leq \delta \leq \min\{\tilde{\delta}^{s+}, 1\}$*, where*

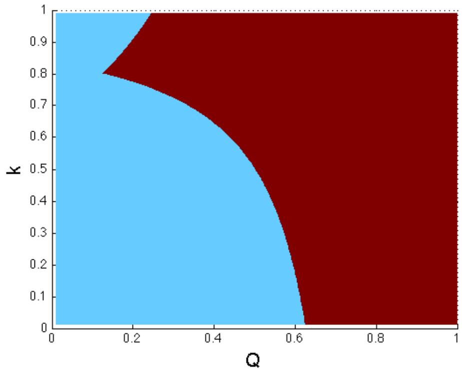
$$\tilde{\delta}^{s+} \equiv \min\left\{ \frac{\bar{a}_{ij} - \hat{\gamma}_i \rho_j}{\omega \bar{a}_{ij} - \|\boldsymbol{\lambda}_i\|^2 \hat{\gamma}_i \rho_j} \middle| i \in \mathcal{I}, j \in \mathcal{J}, \omega \bar{a}_{ij} - \|\boldsymbol{\lambda}_i\|^2 \hat{\gamma}_i \rho_j > 0 \right\} \text{ and} \tag{5.2}$$

$$\tilde{\delta}^{s-} \equiv \max\left\{ \frac{\bar{a}_{ij} - \hat{\gamma}_i \rho_j}{\omega \bar{a}_{ij} - \|\boldsymbol{\lambda}_i\|^2 \hat{\gamma}_i \rho_j} \middle| i \in \mathcal{I}, j \in \mathcal{J}, \omega \bar{a}_{ij} - \|\boldsymbol{\lambda}_i\|^2 \hat{\gamma}_i \rho_j < 0 \right\}, \tag{5.3}$$

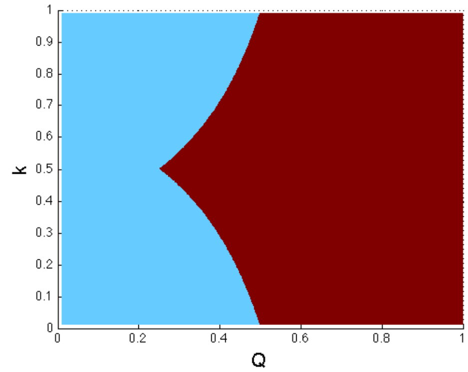
*with the convention that the minimum over an empty set is positive infinity and the maximum over an empty set is negative infinity.*

For markets where $\bar{\mathbf{a}}_i - \hat{\gamma}_i \boldsymbol{\rho} \geq \mathbf{0}$ for all $i \in \mathcal{I}$, myopic behavior will always lead to a solution where all consumer segments own each product. As consumers become more strategic, those who have more (less) relative certainty over their personal dynamics will demand more (less) products. Therefore, as $\delta$ increases, with sufficient market certainty, the threshold $\tilde{\delta}^{s+}$ will be reached and consumers will start to sell entire stocks of their products. However, if $\omega \bar{a}_{ij} - \|\boldsymbol{\lambda}_i\|^2 \hat{\gamma}_i \rho_j \leq 0$ for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$, in addition to $\bar{\mathbf{a}}_i - \hat{\gamma}_i \boldsymbol{\rho} \geq \mathbf{0}$, then $\tilde{\delta}^{s+} \geq 1$ and

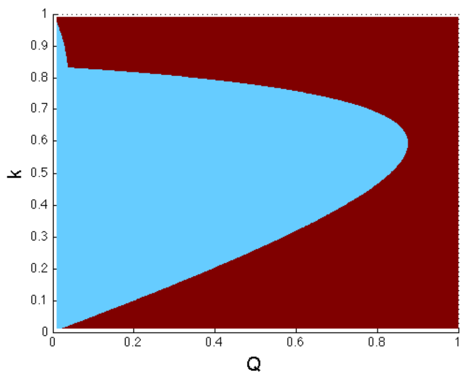
Figure 5: Regions (red) where an equilibrium solution with $\mathbf{s}_i = \mathbf{0}$ exists across different population sizes and total quantity of products.



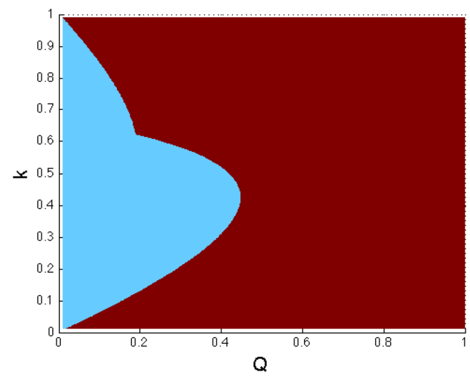

(a) $\gamma_1 = 0.5, \alpha = k$




(b) $\gamma_1 = 1, \alpha = k$




(c) $\gamma_1 = 0.5, \alpha = 0.3$




(d) $\gamma_1 = 1, \alpha = 0.3$

condition (5.1) will hold for all $\delta \in [0, 1)$. For markets where $\bar{\mathbf{a}}_i - \hat{\gamma}_i \boldsymbol{\rho} < \mathbf{0}$ for all $i \in \mathcal{I}$ myopic behavior will always lead to a solution where some consumers sell their entire stock of product, since there is a relative shortage of desired products. If dynamics allow for consumers to want different goods over time, then higher levels of strategic behavior could lead to an equilibrium solution with $\mathbf{s}_i$. We refer to the case of market parameters where $\bar{\mathbf{a}}_i - \hat{\gamma}_i \boldsymbol{\rho} \geq \mathbf{0}$ for all $i$ and $\tilde{\delta}^{s+} \geq 1$ as comprehensive stationary markets for $\delta \in [0, 1)$, since condition (5.1) holds for all feasible $\delta$.

**Example 5.** Consider the type specific framework from Example 3 extended to infinite horizon setting with $a = 0.25$, $b = 0.5$, and $\gamma_2 = 1$. For various values of $\gamma_1$, $\alpha$, $k$, and $Q$, the red areas in Figure 5 shows the parameter values where condition (5.1) holds for all $\delta \in [0, 1)$. To ensure a stationary solution for each value of $k$, $\lambda_1 = 1 - k$ and $\lambda_2 = k$. The example demonstrates that the set of comprehensive stationary markets for $\delta \in [0, 1)$ is non-empty. In each case, there is a nonlinear threshold with respect to the total quantity of products in the market, that determines if the market is comprehensive stationary for $\delta \in [0, 1)$. Since the set is non-empty, we can now characterize the optimal level of strategic behavior for these markets.

**Proposition 6.** *The optimal level of strategic behavior for a comprehensive stationary market over $\delta \in [0, 1)$ is $\delta^* = 0$.*

In a comprehensive stationary markets for $\delta \in [0, 1)$, the product balance is sufficient to guarantee that each consumer owns quantities of each product. Proposition 6 shows that for in these stationary markets, it is optimal for consumers to be myopic. Since the product supply does not cause stock outs, if $\bar{\bar{\mathbf{B}}}^{-1}\rho$ is positive, then increasing the level of strategic behavior leads to greater prices for all products. The consistent result in the infinite horizon case supports the previous result of myopic behavior maximizing aggregate surplus in the two period model.

# 6. Conclusions

In this paper we study a C2C exchange market for an arbitrary number of substitutable products with an arbitrary number of representative consumers, differentiated by population size, product preferences, and price sensitivity. In a static setting, the market clearing price and equilibrium quantities traded are shown to be unique when the representative consumers have correlated price sensitivities. Since consumer preferences for durable products are often time-dependent, we incorporate dynamics into the modeling framework. We use the structure of the the static model to show the uniqueness of the equilibrium in the multi-period case. The results in the dynamic model provide insights into the influence of strategic behavior on prices, quantities traded, and the general surplus of C2C markets.

Given the global growth in C2C trading and the dynamic nature of consumer preferences, understanding the factors that influence prices and the general surplus of participants is becoming increasingly important. By providing a general framework of C2C trading, we hope to expand secondary market research applications. Generalizing the model to incorporate production from either a monopolist or from firms in a competitive environment will allow issues such as leasing versus selling and product durability to be examined in a multi-product setting with consumer dynamics and strategic behavior. Future research can further generalize the model to capture B2B transactions. Incorporating quantity competition amongst producers would broaden the scope of the model to applications involving trade within commodity markets and the movement of goods through supply chains.

# Technical Appendix

## Proof of Proposition 2

Summing (3.3) over all $i \in \mathcal{I}$ and using the market clearing conditions (3.5) implies that $\sum_{i\in\mathcal{I}} \big(\bar{\mathbf{a}}_i - \bar{\mathbf{B}}_i(\mathbf{p} - \mathbf{s}_i) - \mathbf{x}_i\big) = \mathbf{0}$. Thus, the equilibrium price is given by $\mathbf{p} = \bar{\bar{\mathbf{B}}}^{-1}\bigg(\bar{\bar{\mathbf{a}}} - \bar{\bar{\mathbf{x}}} + \sum_{i\in\mathcal{I}} \bar{\mathbf{B}}_i\mathbf{s}_i\bigg) = \bar{\bar{\mathbf{B}}}^{-1}\bigg(\boldsymbol{\rho} + \sum_{i\in\mathcal{I}} \bar{\mathbf{B}}_i\mathbf{s}_i\bigg)$ Substituting the price equation into (3.3) and setting $\mathbf{q}_i = \mathbf{x}_i + \mathbf{y}_i$ implies that

$$
\begin{aligned}
\mathbf{q}_i &= \bar{\mathbf{a}}_i - \bar{\mathbf{B}}_i\bar{\bar{\mathbf{B}}}^{-1}\big(\bar{\bar{\mathbf{a}}} - \bar{\bar{\mathbf{x}}}\big) + \bar{\mathbf{B}}_i\bigg(\mathbf{s}_i - \sum_{i'\in\mathcal{I}} \bar{\bar{\mathbf{B}}}^{-1}\bar{\mathbf{B}}_{i'}\mathbf{s}_{i'}\bigg) \\
&= \bar{\mathbf{a}}_i - \hat{\gamma}_i\boldsymbol{\rho} + \bar{\gamma}_i\mathbf{B}\bigg(\mathbf{s}_i - \sum_{i'\in\mathcal{I}} \hat{\gamma}_{i'}\mathbf{s}_{i'}\bigg).
\end{aligned} \tag{6.1}
$$

Applying the complementarity condition, which follows from substituting $\mathbf{q}_i$ into (3.4) and amalgamating (6.1) for all $i \in \mathcal{I}$ leads to (3.6).

## Preliminary Results for Proof of Theorem 1

Uniqueness under correlated demand sensitivities, results from the connectivity of trade and aggregate preferences, population, and sensitivity properties in the structure of $\mathbf{r}$ and $\mathbf{M}$.

**Lemma 1.** *The matrix* $\mathbf{M}$ *is symmetric positive semi-definite.*

*Proof.* If a symmetric matrix is diagonally dominant with positive diagonal elements, then the matrix is positive semi-definite. Noting that $\mathbf{B}$ has positive diagonals, the sums of the off diagonals are equal in magnitude to the diagonal element of in each row, and $\hat{\gamma}_i\bar{\gamma}_j = \bar{\gamma}_i\hat{\gamma}_j$, the matrix $\mathbf{M}$ is symmetric positive semi-definite. □

**Lemma 2.** *The null space of* $\mathbf{M}$ *consists of all vector* $\mathbf{u} = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_m]$*, with* $\mathbf{u}_i \in \Re^n$*, such that* $\mathbf{u}_1 = \mathbf{u}_2 = \cdots = \mathbf{u}_m$.

*Proof.* The null space of $\mathbf{M}$ is any vector $\mathbf{u} \in \Re^{mn}$ that solves $\mathbf{M}\mathbf{u} = \mathbf{0}$. Defining the matrices $\boldsymbol{\Gamma} = \mathrm{diag}(\bar{\mathbf{B}}_1, \bar{\mathbf{B}}_2, \ldots, \bar{\mathbf{B}}_m)$ and

$$
\tilde{\mathbf{M}} = \begin{bmatrix} (1-\hat{\gamma}_1)\mathbf{I} & -\hat{\gamma}_2\mathbf{I} & \cdots & -\hat{\gamma}_m\mathbf{I} \\ -\hat{\gamma}_1\mathbf{I} & (1-\hat{\gamma}_2)\mathbf{I} & \cdots & -\hat{\gamma}_m\mathbf{I} \\ \vdots & \vdots & \ddots & \vdots \\ -\hat{\gamma}_1\mathbf{I} & -\hat{\gamma}_2\mathbf{I} & \cdots & (1-\hat{\gamma}_m)\mathbf{I} \end{bmatrix},
$$

we can express the matrix $\mathbf{M}$ as $\boldsymbol{\Gamma}\tilde{\mathbf{M}}$. From the invertibility of $\boldsymbol{\Gamma}$, the null space is simplified to $\mathrm{Null}(\mathbf{M}) \equiv \{\mathbf{u} \in \Re^{mn} : \tilde{\mathbf{M}}\mathbf{u} = \mathbf{0}\}$. Consider an $m$-block vector $\mathbf{u} = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_m]$, where $\mathbf{u}_i \in \Re^n$. If $\mathbf{u}$ is in the null space of $\mathbf{M}$, then $\mathbf{u}_i = \sum_{i'\in\mathcal{I}} \hat{\gamma}_{i'}\mathbf{u}_{i'}$ for all $i \in \mathcal{I}$. This implies that $\mathbf{u}_1 = \mathbf{u}_2 = \cdots = \mathbf{u}_m$. Moreover, any vector $\mathbf{u}$ of this form satisfies $\tilde{\mathbf{M}}\mathbf{u} = \mathbf{0}$. □

**Lemma 3.** *The linear complementarity problem (6.1) is feasible.*

*Proof.* By definition the LCP $(\mathbf{r}, \mathbf{M})$ is feasible if and only if there exists vector $\hat{\mathbf{s}} \geq 0$ such that $\mathbf{q} \geq 0$. Rearranging (6.1) and emphasizing that $\mathbf{q}$ is a direct function of multipliers $\mathbf{s}$, it follows that $\mathbf{q}_i(\mathbf{s}) = \bar{\mathbf{a}}_i - \hat{\gamma}_i\bar{\bar{\mathbf{a}}} + \bar{\gamma}_i\mathbf{B}\left(\mathbf{s}_i - \sum_{i' \in \mathcal{I}} \hat{\gamma}_{i'}\mathbf{s}_{i'}\right) + \hat{\gamma}_i\bar{\bar{\mathbf{x}}}$. It is apparent that if there exists a vector $\tilde{\mathbf{s}} \geq \mathbf{0}$ such that $\mathbf{q}_i(\mathbf{s})$ is nonnegative for all $i$ when $\bar{\bar{\mathbf{x}}} = 0$, then the LCP will be feasible for any $\bar{\bar{\mathbf{x}}} \in \Re^n_+$. To show feasibility we construct a vector $\tilde{\mathbf{s}} \geq \mathbf{0}$ for an LCP with $\bar{\bar{x}} = \mathbf{0}$ such that $\mathbf{r} + \mathbf{M}\tilde{\mathbf{s}} \geq 0$.

Define $\bar{\mathbf{r}}_i = \bar{\mathbf{a}}_i - \hat{\gamma}_i\bar{\bar{\mathbf{a}}}$ for all $i \in \mathcal{I}$ and the sub-vector and sub-matrix

$$\hat{\mathbf{r}} = \begin{bmatrix} \bar{\mathbf{r}}_1 \\ \bar{\mathbf{r}}_2 \\ \vdots \\ \bar{\mathbf{r}}_{m-1} \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{M}} = \begin{bmatrix} \bar{\gamma}_1(1-\hat{\gamma}_1)\mathbf{B} & -\bar{\gamma}_1\hat{\gamma}_2\mathbf{B} & \cdots & -\bar{\gamma}_1\hat{\gamma}_{m-1}\mathbf{B} \\ -\bar{\gamma}_2\hat{\gamma}_1\mathbf{B} & \bar{\gamma}_2(1-\hat{\gamma}_2)\mathbf{B} & \cdots & -\bar{\gamma}_2\hat{\gamma}_{m-1}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\gamma}_{m-1}\hat{\gamma}_1\mathbf{B} & -\bar{\gamma}_{m-1}\hat{\gamma}_2\mathbf{B} & \cdots & \bar{\gamma}_{m-1}(1-\hat{\gamma}_{m-1})\mathbf{B} \end{bmatrix}.$$

Noting that $\hat{\mathbf{M}}$ is strictly diagonally dominant, there is a vector $\hat{\mathbf{z}} = \hat{\mathbf{M}}^{-1}\hat{\mathbf{r}}$ which uniquely solves the subsystem, where $\hat{\mathbf{z}} = \begin{bmatrix}\hat{\mathbf{z}}_1 & \hat{\mathbf{z}} & \cdots & \hat{\mathbf{z}}_{m-1}\end{bmatrix}$. Define the vector $\mathbf{r}_m$ and matrix $\mathbf{M}_m$ as $\mathbf{r}_m = \bar{\mathbf{a}}_m - \hat{\gamma}_m\bar{\bar{\mathbf{a}}}$ and $\mathbf{M}_m = \begin{bmatrix} -\bar{\gamma}_m\hat{\gamma}_1\mathbf{B} & -\bar{\gamma}_m\hat{\gamma}_2\mathbf{B} & \cdots & \bar{\gamma}_m(1-\hat{\gamma}_m)\mathbf{B}\end{bmatrix}$, respectively. The vector $\mathbf{z} = [\mathbf{z}_1 \quad \mathbf{z}_2 \quad \cdots \quad \mathbf{z}_{m-1} \quad \mathbf{0}]$, where $\mathbf{z}_i = \hat{\mathbf{z}}_i$ for all $i \in \{1 \ldots m-1\}$, solves the system $\mathbf{Mz} = \mathbf{r}$ if and only if $\mathbf{r}_m = \mathbf{M}_m\mathbf{z} = -\bar{\gamma}_m \sum_{i=1}^{m-1} \hat{\gamma}_i\mathbf{B}\mathbf{z}_i$. Since $\bar{\mathbf{r}}_i = \bar{\gamma}_i\big(\mathbf{B}\mathbf{z}_i - \sum_{j=1}^{m-1}\hat{\gamma}_j\mathbf{B}\mathbf{z}_j\big)$ for $i \in \{1 \ldots m-1\}$, $\bar{\bar{\mathbf{r}}} = \sum_{i \in \mathcal{I}} \bar{\mathbf{r}}_i = \mathbf{0}$, and $\bar{\gamma}_i = \hat{\gamma}_i \sum_{j \in \mathcal{I}} \bar{\gamma}_j$

$$\begin{aligned} \bar{\mathbf{r}}_m &= -\sum_{i=1}^{m-1} \bar{\mathbf{r}}_i \\ &= \sum_{i=1}^{m-1} \bar{\gamma}_i \sum_{j=1}^{m-1} \hat{\gamma}_j\mathbf{B}\mathbf{z}_j - \sum_{i=1}^{m-1} \bar{\gamma}_i\mathbf{B}\mathbf{z}_i \\ &= \sum_{i=1}^{m-1} \bar{\gamma}_i \sum_{j=1}^{m-1} \hat{\gamma}_j\mathbf{B}\mathbf{z}_j - \sum_{i=1}^{m-1} \hat{\gamma}_i \sum_{j=1}^{m} \bar{\gamma}_j\mathbf{B}\mathbf{z}_i \\ &= -\bar{\gamma}_m \sum_{i=1}^{m-1} \hat{\gamma}_i\mathbf{B}\mathbf{z}_i \end{aligned}$$

Thus, $\mathbf{z}$ is a solution to $\mathbf{r} = \mathbf{Mz}$, but The expression $\mathbf{z}$ is not guaranteed to be nonnegative. However, for each product $j$ we can define a vector $\mathbf{u}_0 = \begin{bmatrix}u_1 & u_2 & \cdots & u_m\end{bmatrix}$, where entry $u_j = -\min_i\{z_{ij}\}$. The block vector $\mathbf{u} = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m\end{bmatrix}$, where each vector $\mathbf{u}_i = \mathbf{u}_0$, by Lemma 2 is in the null space of $\mathbf{M}$. Therefore, by construction, the vector $\tilde{\mathbf{s}} = \mathbf{u} - \mathbf{z}$ is a feasible solution to the LCP $(\mathbf{r}, \mathbf{M})$. □

## Proof of Theorem 1

From Corollary 3.5.6 of Pang et al. (1992), if $\mathbf{M} \in \Re^{mn \times mn}$ is adequate and $\mathbf{r} \in \Re^{mn}$ is arbitrary, then if the LCP is feasible, there exist a unique vector $\mathbf{q}^*$ and a vector $\mathbf{s}$ satisfying

$$\mathbf{q}^* = \mathbf{r} + \mathbf{Ms} \geq \mathbf{0}, \qquad \mathbf{s} \geq \mathbf{0}, \qquad \mathbf{q}^{*T}\mathbf{s} = 0,$$

with no guarantees that the vector $\mathbf{s}$ is unique. If $\bar{\mathbf{s}}$ and $\hat{\mathbf{s}}$ are two solutions to the LCP, then $\mathbf{q}^* = \mathbf{r} + \mathbf{M}\bar{\mathbf{s}} = \mathbf{r} + \mathbf{M}\hat{\mathbf{s}} = \mathbf{q}^*$. Symmetric positive semi-definite matrix belong to the class of matrices that are adequate Pang et al. (1992). Since Lemma 3 shows that the LCP is feasible and $\mathbf{M}$ is a positive semi-definite matrix (and thus also adequate), there is a unique solution $\mathbf{y}^* = \mathbf{q}^* - \mathbf{x}$ which produces the equilibrium quantities traded in the exchange market.

Although matrix $\mathbf{M}$ belongs to a class which only guarantees uniqueness of the end allocation quantities $\mathbf{q}$, the following proposition shows that the problem has a solution that is unique. Corollary 3.5.6 of Pang et al. (1992) is provided for an arbitrary vector $\mathbf{r} \in \Re^{mn}$. The structure of the vector $\mathbf{r}$ in the exchange market is not arbitrary, since the sum of the block components of the vector $\mathbf{r}$ is the market quantity vector $\bar{\bar{\mathbf{x}}}$.

Consider any two solutions $\tilde{\mathbf{s}}$ and $\hat{\mathbf{s}}$ to the LCP such that $\mathbf{r} + \mathbf{M}\tilde{\mathbf{s}} = \mathbf{r} + \mathbf{M}\hat{\mathbf{s}} = \mathbf{q}^*$. It follows that there exists a vector $\tilde{\mathbf{u}} \in \text{Null}(\mathbf{M})$ such that

$$\hat{\mathbf{s}} = \tilde{\mathbf{s}} + \tilde{\mathbf{u}}. \tag{6.2}$$

Given a unique solution $\mathbf{q}^*$ from the result of trading, for each product $j \in \mathcal{J}$ there exists at least one consumer $i$ such that $x^*_{ij} > 0$, implying that there is at least one variable $\tilde{s}_{ij} = 0$ and $\hat{s}_{ij} = 0$ for each $j \in \mathcal{J}$. Since $\tilde{\mathbf{u}}$ must have the same entry $\tilde{u}_{ij}$ equalled to some $\tilde{u}_j$ for all consumer groups $i$ (Lemma 2), the only vector in the Null space that can satisfy (6.2) is $\tilde{\mathbf{u}} = \mathbf{0}$. Thus, $\tilde{\mathbf{s}} = \hat{\mathbf{s}}$ equals the unique solution $\mathbf{s}^*$. From the price equation in Proposition 2, it follows that prices are unique and equal to

$$\mathbf{p}^* = \bar{\bar{\mathbf{B}}}^{-1}\big(\boldsymbol{\rho} + \sum_{i \in \mathcal{I}} \bar{\mathbf{B}}_i \mathbf{s}^*_i\big).$$

## Proof of Corollary 3

The solution to LCP $(\mathbf{r}, \mathbf{M})$ does not consider the initial quantity allocation state vector $\mathbf{x}$, rather it only depends on the total quantity for each product. Since the realized quantities after the trade are given by $\mathbf{y}^*_i + \mathbf{x}_i = \bar{\mathbf{a}}_i - \bar{\mathbf{B}}_i\big(\mathbf{p}^* - \mathbf{s}^*_i\big)$ and the optimal price vector is $\mathbf{p}^* = \bar{\bar{\mathbf{B}}}^{-1}\big(\boldsymbol{\rho} + \sum_{i \in \mathcal{I}} \bar{\mathbf{B}}_i \mathbf{s}^*_i\big)$, the end distribution of goods and prices are independent of the initial distribution of products amongst consumers. The aggregate surplus is the sum of the individual surpluses, which is

$$\begin{aligned}\sum_{i \in \mathcal{I}} CS_i(\mathbf{x}_i) &= \sum_{i \in \mathcal{I}} \Bigg( \big(\mathbf{x}_i + \mathbf{y}^*_i\big)^T \bigg( \tilde{\mathbf{a}}_i - \frac{1}{2k_i} \tilde{\mathbf{B}}_i \big(\mathbf{x}_i + \mathbf{y}^*_i\big) \bigg) - \mathbf{y}^{*T}_i \mathbf{p}^* \Bigg) \\ &= \sum_{i \in \mathcal{I}} \Bigg( \big(\bar{\mathbf{a}}_i - \bar{\mathbf{B}}_i\big(\mathbf{p}^* - \mathbf{s}^*_i\big)\big)^T \bigg( \tilde{\mathbf{a}}_i - \frac{1}{2k_i} \tilde{\mathbf{B}}_i \big(\bar{\mathbf{a}}_i - \bar{\mathbf{B}}_i\big(\mathbf{p}^* - \mathbf{s}^*_i\big)\big) \bigg) \Bigg).\end{aligned}$$

Therefore, equilibrium price vector, quantities traded, and aggregate surplus are independent of the individual starting quantities.

## Preliminary Results for Proof of Theorem 2

Before establishing the uniqueness of the multi-period exchange market, it is helpful to present several technical lemmas.

**Lemma 4.** *For $t \in \mathcal{T}$ define the vectors $\mathbf{q}_t$ and $\tilde{\mathbf{r}}_t(\mathbf{p}_{t+1})$ as*

$$\mathbf{q}_t = \begin{bmatrix} \mathbf{x}_{t1} + \mathbf{y}_{t1} \\ \mathbf{x}_{t2} + \mathbf{y}_{t2} \\ \vdots \\ \mathbf{x}_{tm} + \mathbf{y}_{tm} \end{bmatrix} \qquad \text{and} \qquad \tilde{\mathbf{r}}_t(\mathbf{p}_{t+1}) = \begin{bmatrix} \bar{\mathbf{a}}_{t1} - \hat{\gamma}_{t1}\boldsymbol{\rho}_t + \delta\phi_{t1}\bar{\mathbf{B}}_{t1}\mathbf{p}_{t+1} \\ \bar{\mathbf{a}}_{t2} - \hat{\gamma}_{t2}\boldsymbol{\rho}_t + \delta\phi_{t2}\bar{\mathbf{B}}_{t2}\mathbf{p}_{t+1} \\ \vdots \\ \bar{\mathbf{a}}_{tm} - \hat{\gamma}_{tm}\boldsymbol{\rho}_t + \delta\phi_{tm}\bar{\mathbf{B}}_{tm}\mathbf{p}_{t+1} \end{bmatrix},$$

*and the block matrix $\mathbf{M}_t$ as*

$$\mathbf{M}_t = \begin{bmatrix} \bar{\gamma}_{t1}(1-\hat{\gamma}_{t1})\mathbf{B} & -\bar{\gamma}_{t1}\hat{\gamma}_{t2}\mathbf{B} & \cdots & -\bar{\gamma}_{t1}\hat{\gamma}_{tm}\mathbf{B} \\ -\bar{\gamma}_{t2}\hat{\gamma}_{t1}\mathbf{B} & \bar{\gamma}_{t2}(1-\hat{\gamma}_{t2})\mathbf{B} & \cdots & -\bar{\gamma}_{t2}\hat{\gamma}_{tm}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\gamma}_{tm}\hat{\gamma}_{t1}\mathbf{B} & -\bar{\gamma}_{tm}\hat{\gamma}_{t2}\mathbf{B} & \cdots & \bar{\gamma}_{tm}(1-\hat{\gamma}_{tm})\mathbf{B} \end{bmatrix}.$$

*Given any vector $\mathbf{p}_{t+1}$, the following mixed LCP*

$$\bar{\mathbf{a}}_{ti} - \bar{\mathbf{B}}_{ti}\big(\mathbf{p}_t - \mathbf{s}_{ti} - \delta\|\boldsymbol{\lambda}_i\|^2\mathbf{p}_{t+1}\big) - \mathbf{x}_{ti} - \mathbf{y}_{ti} = \mathbf{0}, \quad \forall i \in \mathcal{I} \tag{6.3}$$

$$\mathbf{0} \le \mathbf{s}_i \perp \mathbf{x}_i + \mathbf{y}_i \ge \mathbf{0}, \quad \forall i \in \mathcal{I} \tag{6.4}$$

$$\sum_{i\in\mathcal{I}} \mathbf{y}_{ti} = \mathbf{0} \tag{6.5}$$

*is equivalent to finding a vector $\mathbf{s}_t$ which solves the LCP*

$$\mathbf{0} \le \mathbf{s}_t \perp \mathbf{q}_t = \tilde{\mathbf{r}}_t(\mathbf{p}_{t+1}) + \mathbf{M}_t\mathbf{s}_t \ge \mathbf{0}. \tag{6.6}$$

*resulting in prices*

$$\mathbf{p}_t = \bar{\bar{\mathbf{B}}}_t^{-1}\boldsymbol{\rho}_t - \sum_{i\in\mathcal{I}} \hat{\gamma}_{ti}\big(\mathbf{s}_{ti} - \delta\|\boldsymbol{\lambda}_i\|^2\mathbf{p}_{t+1}\big). \tag{6.7}$$

*Proof.* Summing the demand equation (6.3) over all $i \in \mathcal{I}$, applying market clearing condition (6.5), and isolating for price implies that the fictitious agent will select

$$\begin{aligned} \mathbf{p}_t &= \bar{\bar{\mathbf{B}}}_t^{-1}\left(\bar{\bar{\mathbf{a}}}_t - \bar{\bar{\mathbf{x}}} - \sum_{i\in\mathcal{I}} \bar{\mathbf{B}}_{ti}\left(\mathbf{s}_{ti} - \delta\sum_{j\in\mathcal{I}}\lambda_{ij}^2\mathbf{p}_{t+1}\right)\right) \\ &= \bar{\bar{\mathbf{B}}}_t^{-1}\boldsymbol{\rho}_t - \sum_{i\in\mathcal{I}} \hat{\gamma}_{ti}\big(\mathbf{s}_{ti} - \delta\|\boldsymbol{\lambda}_i\|^2\mathbf{p}_{t+1}\big) \\ &= \bar{\bar{\mathbf{B}}}_t^{-1}\boldsymbol{\rho}_t + \sum_{i\in\mathcal{I}} \hat{\gamma}_{ti}\mathbf{s}_{ti}^* + \delta\omega_t\mathbf{p}_{t+1}^* \end{aligned}$$

Substituting $\mathbf{q}_{ti} = \mathbf{x}_{ti} + \mathbf{y}_{ti}$ and the market clearing price $\mathbf{p}_t$ into (6.3) implies that the end product allocations in equilibrium in period $t$ is

$$\begin{aligned} \mathbf{q}_{ti} &= \bar{\mathbf{a}}_{ti} - \bar{\mathbf{B}}_{ti}\left(\bar{\bar{\mathbf{B}}}_t^{-1}\boldsymbol{\rho}_t + \sum_{i\in\mathcal{I}}\hat{\gamma}_{ti}\mathbf{s}_{ti}^* + \delta\omega_t\mathbf{p}_{t+1}^*\right) - \mathbf{s}_{ti} - \delta\|\boldsymbol{\lambda}_i\|^2\mathbf{p}_{t+1}\Bigg) \\ &= \bar{\mathbf{a}}_{ti} - \hat{\gamma}_{ti}\boldsymbol{\rho}_t + \delta\phi_{ti}\bar{\mathbf{B}}_{ti}\mathbf{p}_{t+1} + \bar{\mathbf{B}}_{ti}\left(\mathbf{s}_{ti} - \sum_{i'\in\mathcal{I}}\hat{\gamma}_{ti'}\mathbf{s}_{ti'}\right). \end{aligned}$$

The complementarity condition follows from substituting $\mathbf{q}_t$ into (6.4). □

**Lemma 5.** *The complementarity problem (6.6) is feasible for any vector* $\mathbf{p}_{t+1}$.

*Proof.* Defining $\bar{\mathbf{r}}_{ti}(\mathbf{p}_{t+1}) = \bar{\mathbf{a}}_{ti} - \hat{\gamma}_{ti}\bar{\bar{\mathbf{a}}} + \delta\phi_{ti}\bar{\mathbf{B}}_{ti}\mathbf{p}_{t+1}$ for all $i \in \mathcal{I}$, the steps in the proof of Proposition 3 will guarantee feasibility of the LCP ($\tilde{\mathbf{r}}_t(\mathbf{p}_{t+1})$,$\mathbf{M}_t$) if $\sum_{i\in\mathcal{I}} \bar{\mathbf{r}}_{ti}(\mathbf{p}_{t+1}) = \mathbf{0}$. Since $\sum_{i\in\mathcal{I}} \left(\bar{\mathbf{a}}_{ti} - \hat{\gamma}_{ti}\bar{\bar{\mathbf{a}}}\right) = \mathbf{0}$, it follows that

$$
\begin{aligned}
\sum_{i\in\mathcal{I}} \bar{\mathbf{r}}_{ti}(\mathbf{p}_{t+1}) &= \sum_{i\in\mathcal{I}} \delta\phi_{ti}\bar{\mathbf{B}}_{ti}\mathbf{p}_{t+1} \\
&= \left(\sum_{i\in\mathcal{I}} \delta\phi_{ti}\bar{\gamma}_{ti}\right)\mathbf{B}\mathbf{p}_{t+1} \\
&= \delta\left(\sum_{i\in\mathcal{I}} \bar{\gamma}_{ti}\|\boldsymbol{\lambda}_i\|^2 - \sum_{i\in\mathcal{I}} \bar{\gamma}_{ti} \sum_{i'\in\mathcal{I}} \hat{\gamma}_{ti'}\|\boldsymbol{\lambda}_{i'}\|^2\right)\mathbf{B}\mathbf{p}_{t+1} \\
&= \delta\left(\sum_{i\in\mathcal{I}} \hat{\gamma}_{ti} \sum_{i'\in\mathcal{I}} \bar{\gamma}_{ti'}\|\boldsymbol{\lambda}_i\|^2 - \sum_{i\in\mathcal{I}} \bar{\gamma}_{ti} \sum_{i'\in\mathcal{I}} \hat{\gamma}_{ti'}\|\boldsymbol{\lambda}_{i'}\|^2\right)\mathbf{B}\mathbf{p}_{t+1} \\
&= \mathbf{0},
\end{aligned}
$$

and the LCP ($\mathbf{r}_t(\mathbf{p}_{t+1})$,$\mathbf{M}_t$) is feasible for any $\mathbf{p}_{t+1}$. □

## Proof of Theorem 2

To prove the theorem we use a three step induction process, making inductive assumptions on claims 1, 2 and 4. We start by noting that $v_{T+1,i} = 0$ and $\mathbf{p}_{T+1} = \mathbf{0}$. This implies that in period $T$, the problem is equivalent to the static model. Following the results of Section 3 and defining $\mathbf{r}_T = \tilde{\mathbf{r}}_T(\mathbf{0})$, the first order conditions of the optimization problem with market clearing leads to the LCP $(\mathbf{r}_T, \mathbf{M}_T)$. Since $\mathbf{M}_T$ is symmetric positive semi-definite and the complementarity problem is feasible (Lemma 5), the solution to the LCP, $\mathbf{s}^*_T$, is unique and independent of the state $\mathbf{x}_T$. This establishes the base of the inductive assumption on independence of multiplier $\mathbf{s}^*_t$ from state $\mathbf{x}_T$. Theorem 1 the price $\mathbf{p}^*_T = \bar{\bar{\mathbf{B}}}_T^{-1}\left(\bar{\bar{\mathbf{a}}}_T - \bar{\bar{\mathbf{x}}} + \sum_{i\in\mathcal{I}} \bar{\mathbf{B}}_{Ti}\mathbf{s}^*_{Ti}\right)$ is then also unique and independent of the state $\mathbf{x}_T$ and trivially fits the expressions given by (4.5) and also serves as a base for the inductive assumption on state independence of prices. The resulting quantity of products traded by consumer $i$ is $\mathbf{y}^*_{Ti} = \bar{\mathbf{a}}_{Ti} - \mathbf{x}_{Ti} - \hat{\gamma}_{Ti}\boldsymbol{\rho}_T + \bar{\mathbf{B}}_{Ti}\left(\mathbf{s}^*_{Ti} - \sum_{i'\in\mathcal{I}} \hat{\gamma}_{Ti'}\mathbf{s}^*_{Ti'}\right)$, which also trivially matches (4.6) for period $T$. The value function in period $T-1$ is

$$
\begin{aligned}
v_{(T-1)i}(\mathbf{x}_{(T-1)}, \mathbf{p}_{(T-1)}) = \max_{\mathbf{q}_{(T-1)i}\geq\mathbf{0}} \quad & \mathbf{q}^T_{(T-1)i}\left(\tilde{\mathbf{a}}_i - \frac{1}{2k_{(T-1)i}}\tilde{\mathbf{B}}_i\mathbf{q}_{(T-1)i}\right) - \mathbf{y}^T_{(T-1)i}\mathbf{p}_{T-1} && (6.8) \\
& + \delta\sum_{j\in\mathcal{I}} \lambda_{ij}\left(c_{Tj} + \mathbf{x}^T_{Ti}\mathbf{p}^*_T\right) && (6.9)
\end{aligned}
$$

where $c_{Tj} = \mathbf{q}_{Tj}^T\left(\tilde{\mathbf{a}}_j - \frac{1}{2k_{Tj}}\tilde{\mathbf{B}}_j\mathbf{q}_{Tj}^T\right) - \mathbf{q}_{Tj}^T\mathbf{p}_T$ is a constant dependent upon input parameters. Applying the product allocation dynamic constraint constraint (4.3) to (6.10) implies that

$$v_{(T-1)i}(\mathbf{x}_{(T-1)}, \mathbf{p}_{(T-1)}) = \max_{\mathbf{q}_{(T-1)i} \geq \mathbf{0}} \quad \mathbf{q}_{(T-1)i}^T\left(\tilde{\mathbf{a}}_i - \frac{1}{2k_{(T-1)i}}\tilde{\mathbf{B}}_i\mathbf{q}_{(T-1)i}\right) - \mathbf{y}_{(T-1)i}^T\mathbf{p}_{T-1} + \delta\sum_{j\in\mathcal{I}}\lambda_{ij}\left(c_{Tj} + \sum_{j'\in\mathcal{I}}\lambda_{j'j}\big(\mathbf{x}_{(T-1)j'} + \mathbf{y}_{(T-1)j'}\big)^T\mathbf{p}_T^*\right). \tag{6.10}$$

Observe that (6.10) matches the expression (4.7) for period $T-1$ after substituting $\mathbf{q}_{(T-1)i}$ with $\mathbf{x}_{(T-1)i} + \mathbf{y}_{(T-1)i}$.

We now make three inductive assumptions starting in some period $t \in \{2, T-1\}$. We assume that the multipliers $\mathbf{s}_\tau^*$ and prices $\mathbf{p}_\tau^*$ are unique and independent of state $\mathbf{x}_\tau$ for $\tau > t$, and we assume that the value-to-go for consumer $j$ is

$$v_{tj}(\mathbf{x}_t, \mathbf{p}_t) = \max_{\mathbf{y}_{tj} > \mathbf{x}_{tj}} \quad \big(\mathbf{x}_{tj} + \mathbf{y}_{tj}\big)^T\left(\tilde{\mathbf{a}}_j - \frac{1}{2k_{tj}}\tilde{\mathbf{B}}_j\big(\mathbf{x}_{tj} + \mathbf{y}_{tj}\big)\right) - \mathbf{y}_{tj}^T\mathbf{p}_t + \delta\sum_{k\in\mathcal{I}}\lambda_{jk}\left(c_{(t+1)k} + \sum_{k'\in\mathcal{I}}\lambda_{k'k}\big(\mathbf{x}_{tk'} + \mathbf{y}_{tk'}\big)^T\mathbf{p}_{t+1}^*\right), \tag{6.11}$$

where $c_{(t+1)k}$ is a constant entirely determined by input parameters. The first order conditions for player $j$ is then

$$\tilde{\mathbf{a}}_j - \frac{1}{k_{tj}}\tilde{\mathbf{B}}_j\big(\mathbf{x}_{tj} + \mathbf{y}_{tj}\big) - \mathbf{p}_t + \mathbf{s}_{tj} + \delta\sum_{k\in\mathcal{I}}\lambda_{jk}^2\mathbf{p}_{t+1}^* = \mathbf{0}. \tag{6.12}$$

Multiplying (6.12) by $k_{tj}$, $\tilde{\mathbf{B}}_j^{-1}$, and noting that $\sum_{k\in\mathcal{I}}\lambda_{jk}^2 = \|\boldsymbol{\lambda}_j\|^2$ and $\bar{\mathbf{a}}_{tj} = k_{tj}\tilde{\mathbf{B}}_j^{-1}\tilde{\mathbf{a}}_j$ leads to the complementarity problem for end quantities owned based on $\mathbf{p}_t$ as

$$\mathbf{0} \leq \mathbf{s}_{tj} \perp \mathbf{x}_{tj} + \mathbf{y}_{tj} = \bar{\mathbf{a}}_{tj} - \bar{\mathbf{B}}_{tj}\big(\mathbf{p}_t - \mathbf{s}_{tj} - \delta\|\boldsymbol{\lambda}_j\|^2\mathbf{p}_{t+1}^*\big) \geq \mathbf{0} \quad \forall j \in \mathcal{I} \tag{6.13}$$

Adding the market clearing condition

$$\sum_{i\in\mathcal{I}}\mathbf{y}_{ti} = \mathbf{0} \tag{6.14}$$

and defining $\mathbf{r}_t = \tilde{\mathbf{r}}_t(\mathbf{p}_{t+1}^*)$, the LCP (6.13)-(6.14) is equivalent to LCP $(\mathbf{r}_t, \mathbf{M}_t)$, where Lemma 4 provides the definition of the LCP $(\mathbf{r}_t, \mathbf{M}_t)$ and the support for the claim. Since the LCP $(\mathbf{r}_t, \mathbf{M}_t)$ is feasible, $\mathbf{M}_t$ is symmetric positive semi-definite, and both $\mathbf{r}_t$ and $\mathbf{M}_t$ are independent of the state $\mathbf{x}_t$, the solution $\mathbf{s}_t^*$ is unique and independent of the state $\mathbf{x}_t$, proving claim 1. From Lemma 4, $\mathbf{p}_t^* = \bar{\bar{\mathbf{B}}}_t^{-1}\boldsymbol{\rho}_t + \sum_{i\in\mathcal{I}}\hat{\gamma}_{ti}\mathbf{s}_{ti}^* + \delta\omega_t\mathbf{p}_{t+1}^*$, which is unique and independent of $\mathbf{x}_t$, proving claim 2. From the definition of $\mathbf{r}_t$, $\mathbf{y}_{tj} = \bar{\mathbf{a}}_{tj} - \mathbf{x}_{tj} - \hat{\gamma}_{tj}\boldsymbol{\rho}_{t1} + \delta\phi_{tj}\bar{\mathbf{B}}_{tj}\mathbf{p}_{t+1}^* + \bar{\mathbf{B}}_{tj}\left(\mathbf{s}_{tj}^* - \sum_{j'\in\mathcal{I}}\hat{\gamma}_{tj'}\mathbf{s}_{tj'}^*\right)$, leading to claim 3. Finally, substituting the expression for $\mathbf{x}_{tj} + \mathbf{y}_{tj}$ into (6.11) implies that

$v_{tj}(\mathbf{x}_t, \mathbf{p}) = \tilde{c}_{tj} + \mathbf{x}_{tj}^T \mathbf{p}_t = \tilde{c}_{tj} + \sum_{j' \in \mathcal{I}} \lambda_{j'j} \big( \mathbf{x}_{(t-1)j'} + \mathbf{y}^*_{(t-1)j'} \big)^T \mathbf{p}_t$, where $\tilde{c}_{t,j}$ is a constant. Therefore,

$$v_{(t-1)i}(\mathbf{x}_{t-1}, \mathbf{p}_{t-1}) = \max_{\mathbf{y}_{(t-1)i} \geq -\mathbf{x}_{(t-1)i}} \quad \big(\mathbf{x}_{(t-1)i} + \mathbf{y}_{(t-1)i}\big)^T \left( \tilde{\mathbf{a}}_i - \frac{1}{2k_{(t-1)i}} \tilde{\mathbf{B}}_i \big( \mathbf{x}_{(t-1)i} + \mathbf{y}_{(t-1)i} \big) \right)$$
$$- \mathbf{y}_{(t-1)i}^T \mathbf{p}_{t-1} + \delta \sum_{j \in \mathcal{I}} \lambda_{ij} \left( \tilde{c}_{tj} + \sum_{j' \in \mathcal{I}} \lambda_{j'j} \big( \mathbf{x}_{(t-1)j'} + \mathbf{y}_{(t-1)j'} \big)^T \mathbf{p}^*_t \right),$$

and the third inductive assumption holds, proving claim 4.

Using the logic of the proof to Corollary 3, to prove claim 5, we simply have to show that the future component of the sum of each player's value function is independent of the starting quantities $\mathbf{x}_t$ in all periods $t$. Both the optimal price vector $\mathbf{p}^*_{t+1}$ and the end allocation vector $\mathbf{q}_{ti} = \mathbf{x}_{ti} + \mathbf{y}_{ti}$ are independent of starting quantities in each period, but the value of $\mathbf{y}_{ti}$ is dependent on state $\mathbf{x}_t$. Therefore, the only impact that the initial allocations is in the linear term $\mathbf{y}_{ti}^T \mathbf{p}^*_{t+1}$. However, from the market clearing constraint, $\sum_{i \in \mathcal{I}} \mathbf{y}_{ti}^T \mathbf{p}^*_{t+1} = 0$, and thus the sum of the future component of the total surplus is also independent of the individual starting quantities.

## Proof of Corollary 4

The price is non-decreasing for product $j$ between periods if and only if

$$\big( \bar{\bar{\mathbf{B}}}_t^{-1} \boldsymbol{\rho}_t \big)_j + \sum_{i \in \mathcal{I}} \hat{\gamma}_{ti} s^*_{tij} + \delta \omega_t p^*_{(t+1)j} \geq p^*_{(t+1)j}. \tag{6.15}$$

If $\mathbf{s}_{tij} = 0$ for all $i \in \mathcal{I}$, the price is non-increasing if and only if $\delta \geq \tilde{\delta}^p_{tj} = \frac{p^*_{(t+1)j} - \big( \bar{\bar{\mathbf{B}}}_t^{-1} \boldsymbol{\rho}_t \big)_j}{\omega_t p^*_{(t+1)j}}$, and is always decreasing for $\delta < \tilde{\delta}^p_{tj}$, proving the second claim. If $\mathbf{s}_{tij} > 0$ for some $i \in \mathcal{I}$, then the condition that $\delta \geq \tilde{\delta}^p_{tj}$ is sufficient to satisfy inequality (6.15). Thus, price is non-increasing for $\delta > \tilde{\delta}^p_{tj}$ proving the first claim.

## Proof of Proposition 3

If $\mathbf{r}_t > \mathbf{0}$, then the LCP $(\mathbf{r}_t, \mathbf{M}_t)$ has a unique solution with $\mathbf{s} = 0$. Define the matrices $\boldsymbol{\Gamma}_t = \text{diag}(\bar{\mathbf{B}}_{t1}, \bar{\mathbf{B}}_{t2}, \ldots, \bar{\mathbf{B}}_{tm})$. If $\boldsymbol{\Gamma}_t^{-1} \mathbf{r}_t > \mathbf{0}$, then $\mathbf{r}_t > \mathbf{0}$, since $\boldsymbol{\Gamma}_t^{-1}$ is positive definite. Therefore,

$$\boldsymbol{\Gamma}_t^{-1} \mathbf{r}_t = \begin{bmatrix} \mathbf{B}_{t1}^{-1} \big( \bar{\mathbf{a}}_{t1} - \hat{\gamma}_{t1} \boldsymbol{\rho}_t + \delta \phi_{t1} \bar{\mathbf{B}}_{t1} \mathbf{p}^*_{t+1} \big) \\ \mathbf{B}_{t2}^{-1} \big( \bar{\mathbf{a}}_{t2} - \hat{\gamma}_{t2} \boldsymbol{\rho}_t + \delta \phi_{t2} \bar{\mathbf{B}}_{t2} \mathbf{p}^*_{t+1} \big) \\ \vdots \\ \mathbf{B}_{tm}^{-1} \big( \bar{\mathbf{a}}_{tm} - \hat{\gamma}_{tm} \boldsymbol{\rho}_t + \delta \phi_{tm} \bar{\mathbf{B}}_{tm} \mathbf{p}^*_{t+1} \big) \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{a}}_{t1} - \bar{\bar{\mathbf{B}}}_t^{-1} \boldsymbol{\rho}_t + \delta \phi_{t1} \mathbf{p}^*_{t+1} \\ \tilde{\mathbf{a}}_{t2} - \bar{\bar{\mathbf{B}}}_t^{-1} \boldsymbol{\rho}_t + \delta \phi_{t2} \mathbf{p}^*_{t+1} \\ \vdots \\ \tilde{\mathbf{a}}_{tm} - \bar{\bar{\mathbf{B}}}_t^{-1} \boldsymbol{\rho}_t + \delta \phi_{tm} \mathbf{p}^*_{t+1} \end{bmatrix},$$

which follows from the fact that $\mathbf{B}_{ti}^{-1} \bar{\mathbf{a}}_{ti} = \tilde{\mathbf{a}}_{ti}$ and $\hat{\gamma}_{ti} \bar{\mathbf{B}}_{ti}^{-1} = \frac{\bar{\gamma}_{ti} \mathbf{B}^{-1}}{\bar{\gamma}_{ti} \sum_{j \in \mathcal{I}} \bar{\gamma}_{tj}} = \frac{\mathbf{B}^{-1}}{\sum_{j \in \mathcal{I}} \bar{\gamma}_{tj}} = \bar{\bar{\mathbf{B}}}_t^{-1}$. If $\tilde{\mathbf{a}}_{ti} - \bar{\bar{\mathbf{B}}}_t^{-1} \boldsymbol{\rho}_t + \delta \phi_{ti} \bar{\mathbf{B}}_{ti} \mathbf{p}^*_{t+1}$ for all $i$, then the unique solution to the LCP is $\mathbf{s}_t = \mathbf{0}$. If condition (4.8) holds for all $\delta \in [0, 1]$ in period 1, then $\tilde{\delta}^{s-} = 0$ and $\tilde{\delta}^{s+} = 1$ and the solution has $\mathbf{s}_t = \mathbf{0}$ for all $\delta$ on the interval $[0, 1]$.

## Proof of Proposition 4

In a two period model, the second period prices is independent of $\delta$ and $\mathbf{x}_1$ (by Theorem 2). The strategic behavior $\delta$ impacts the market clearing prices and quantity decision in the first period. With $\mathbf{s}_1 = \mathbf{0}$ the surplus and quantities traded in period 1 for consumer $i$ in period $t$ are

$$\begin{aligned}\tilde{v}_{1i}^*(\mathbf{x}_1, \mathbf{q}_1, \delta) &= \mathbf{q}_{1i}^{*T}\left(\tilde{\mathbf{a}}_i - \frac{1}{2k_{1i}}\tilde{\mathbf{B}}_i\mathbf{q}_{1i}^*\right) - \left(\mathbf{q}_{1i}^* - \mathbf{x}_{1i}\right)^T\left(\bar{\bar{\mathbf{B}}}_1^{-1}\boldsymbol{\rho}_1 + \delta\omega_1\mathbf{p}_2^*\right)\\ \mathbf{q}_{1i}(\delta) &= \bar{\mathbf{a}}_{1i} - \hat{\gamma}_{1i}\boldsymbol{\rho}_1 + \delta\phi_{1i}\bar{\mathbf{B}}_{1i}\mathbf{p}_2^*\end{aligned}$$

We can separate the surplus of consumer $i$ into the utility segment $\tilde{v}_{1i}^{u*}(\mathbf{q}) = \mathbf{q}_{1i}^{*T}\left(\tilde{\mathbf{a}}_i - \frac{1}{2k_{1i}}\tilde{\mathbf{B}}_i\mathbf{q}_{1i}^*\right)$ and the price segment $\tilde{v}_{1i}^{p*}(\mathbf{x}_1, \mathbf{q}_1, \delta) = -\left(\mathbf{q}_{1i}^* - \mathbf{x}_{1i}\right)^T\left(\bar{\bar{\mathbf{B}}}_1^{-1}\boldsymbol{\rho}_1 + \delta\omega_1\mathbf{p}_2^*\right)$. Since $\sum_{i\in\mathcal{I}}\tilde{v}_{1i}^{p*}(\mathbf{x}_1, \mathbf{q}_1, \delta) = 0$, due to the market clearing mechanism which ensure that $\sum_{i\in\mathcal{I}}\left(\mathbf{q}_{1i}^* - \mathbf{x}_{1i}\right) = \mathbf{0}$, the aggregate surplus is $\bar{\bar{v}}_1(\delta) = \sum_{i\in\mathcal{I}}\tilde{v}_{1i}^{u*}(\delta, \mathbf{q}_i(\delta))$. The first derivative of surplus with respect to $\delta$ is

$$\frac{d}{d\delta}\bar{\bar{v}}_1^*(\delta) = \sum_{i\in\mathcal{I}}\frac{d}{d\delta}\tilde{v}_{1i}^{u*}(\mathbf{q}_{1i}(\delta)) = \sum_{i\in\mathcal{I}}\left(\nabla_{\mathbf{q}_{1i}^*}\tilde{v}_{1i}^{u*}(\mathbf{q}_{1i}(\delta))\right)^T\frac{\partial}{\partial\delta}\mathbf{q}_{1i}(\delta).$$

With respect to the first period utility component of consumer $i$, the gradients has the form:

$$\begin{aligned}\nabla_{\mathbf{q}_{1i}^*}\tilde{v}_{1i}^{u*}(\mathbf{q}_{1i}(\delta)) &= \tilde{\mathbf{a}}_i - \bar{\mathbf{B}}_{1i}^{-1}\mathbf{q}_{1i}^*\\ &= \tilde{\mathbf{a}}_i - \bar{\mathbf{B}}_{1i}^{-1}\left(\bar{\mathbf{a}}_{1i} - \hat{\gamma}_{1i}\boldsymbol{\rho}_1 + \delta\phi_{1i}\bar{\mathbf{B}}_{1i}\mathbf{p}_2^*\right)\\ &= \tilde{\mathbf{a}}_i - \bar{\mathbf{B}}_{1i}^{-1}\bar{\mathbf{a}}_{1i} + \hat{\gamma}_{1i}\bar{\mathbf{B}}_{1i}^{-1}\boldsymbol{\rho}_1 - \delta\phi_{1i}\mathbf{p}_2^*\\ &= \hat{\gamma}_{1i}\bar{\mathbf{B}}_{1i}^{-1}\boldsymbol{\rho}_1 - \delta\phi_{1i}\mathbf{p}_2^*\\ &= \bar{\bar{\mathbf{B}}}_1^{-1}\boldsymbol{\rho}_1 - \delta\phi_{1i}\mathbf{p}_2^*\end{aligned}$$

where the last inequality follows from the fact that $\hat{\gamma}_{1i}\bar{\mathbf{B}}_{1i}^{-1} = \frac{\bar{\gamma}_{1i}\mathbf{B}^{-1}}{\bar{\gamma}_{1i}\sum_{j\in\mathcal{I}}\bar{\gamma}_{1j}} = \frac{\mathbf{B}^{-1}}{\sum_{j\in\mathcal{I}}\bar{\gamma}_{1j}} = \bar{\bar{\mathbf{B}}}_1^{-1}$. Since $\frac{\partial}{\partial\delta}\mathbf{q}_{1i}(\delta) = \phi_{1i}\bar{\mathbf{B}}_{1i}\mathbf{p}_2^*$, it follows that

$$\nabla_{\mathbf{q}_{1i}^*}\tilde{v}_{1i}^{u*}(\mathbf{q}_{1i}(\delta))^T\frac{\partial}{\partial\delta}\mathbf{q}_{1i}(\delta) = \left(\bar{\bar{\mathbf{B}}}_1^{-1}\boldsymbol{\rho}_1 - \delta\phi_{1i}\mathbf{p}_2^*\right)^T\phi_{1i}\bar{\mathbf{B}}_{1i}\mathbf{p}_2^* = \left(\bar{\bar{\mathbf{B}}}_1^{-1}\boldsymbol{\rho}_1 - \delta\phi_{1i}\mathbf{p}_2^*\right)^T\phi_{1i}\bar{\gamma}_{1i}\mathbf{B}\mathbf{p}_2^*$$

and

$$\sum_{i\in\mathcal{I}}\left(\nabla_{\mathbf{q}_{1i}^*}\tilde{v}_{1i}^{u*}(\mathbf{q}_{1i}(\delta))\right)^T\frac{\partial}{\partial\delta}\mathbf{q}_{1i}(\delta) = \sum_{i\in\mathcal{I}}\phi_{1i}\bar{\gamma}_{1i}\left(\bar{\bar{\mathbf{B}}}_1^{-1}\boldsymbol{\rho}_1\right)^T\mathbf{B}\mathbf{p}_2^* - \delta\sum_{i\in\mathcal{I}}\phi_{1i}^2\bar{\gamma}_{1i}\mathbf{p}_2^{*T}\mathbf{B}\mathbf{p}_2^*.$$

Multiplying and dividing the term $\sum_{i\in\mathcal{I}}\phi_{1i}\bar{\gamma}_{1i}\left(\bar{\bar{\mathbf{B}}}_1^{-1}\boldsymbol{\rho}_1\right)^T\mathbf{B}\mathbf{p}_2^*$ by $\bar{\bar{\gamma}}$, leads to $\bar{\bar{\gamma}}\sum_{i\in\mathcal{I}}\phi_{1i}\hat{\gamma}_{1i}\left(\bar{\bar{\mathbf{B}}}_1^{-1}\boldsymbol{\rho}_1\right)^T\mathbf{B}\mathbf{p}_2^* = 0$, since $\sum_{i\in\mathcal{I}}\phi_{1i}\hat{\gamma}_{1i} = 0$, and $\delta^* = 0$.

## Proof of Proposition 5

We look for stationary solution with $\mathbf{s}_i = \mathbf{0}$. Since the system is in steady state, for convenience we drop the time index for all input parameters. Recall that $\omega = \sum_{i\in\mathcal{I}}\hat{\gamma}_i\|\boldsymbol{\lambda}_i\|^2$. Therefore, $\mathbf{p}^* = \bar{\bar{\mathbf{B}}}^{-1}\boldsymbol{\rho} + \delta\omega\mathbf{p}^*$, which implies that if a stationary solution exists, $\mathbf{p}^* = \frac{1}{1-\delta\omega}\bar{\bar{\mathbf{B}}}^{-1}\boldsymbol{\rho}$. From the optimal quantities exchanged and complementarity condition given by (6.13), there exists a steady state solution with zero multipliers if $\bar{a}_{ij} - \hat{\gamma}_i\rho_j + \frac{\delta\phi_i\hat{\gamma}_i}{1-\delta\sum_{i'\in\mathcal{I}}\hat{\gamma}_{i'}\|\boldsymbol{\lambda}_{i'}\|^2}\rho_j \geq 0$ for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$. Multiplying the inequality by $1-\delta\omega$ leads to the inequality

$$\begin{aligned}\big(1-\delta\omega\big)\big(\bar{a}_{ij} - \hat{\gamma}_i\rho_j\big) + \delta\phi_i\hat{\gamma}_i\rho_j &= \bar{a}_{ij} - \hat{\gamma}_i\rho_j - \delta\big(\omega\big(\bar{a}_{ij} - \hat{\gamma}_i\rho_j\big) - \phi_i\hat{\gamma}_i\rho_j\big)\\ &= \bar{a}_{ij} - \hat{\gamma}_i\rho_j - \delta\big(\omega\bar{a}_{ij} - \omega\hat{\gamma}_i\rho_j - \big(\|\boldsymbol{\lambda}_i\|^2 - \omega\big)\hat{\gamma}_i\rho_j\big)\\ &= \bar{a}_{ij} - \hat{\gamma}_i\rho_j - \delta\big(\omega\bar{a}_{ij} - \|\boldsymbol{\lambda}_i\|^2\hat{\gamma}_i\rho_j\big) \geq 0\end{aligned}$$

Thus, if $\bar{a}_{ij} - \hat{\gamma}_i\rho_j - \delta\big(\omega\bar{a}_{ij} - \|\boldsymbol{\lambda}_i\|^2\hat{\gamma}_i\rho_j\big) \geq 0$ is satisfied for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$, then a steady state solution with $\mathbf{s}_i = \mathbf{0}$ exist. If $\mathbf{s}_i = 0$, then once the system reaches a steady state, $\mathbf{p}^* = \frac{1}{1-\delta\omega}\bar{\bar{\mathbf{B}}}^{-1}\boldsymbol{\rho}$ and $\mathbf{q}_i^* = \bar{\mathbf{a}}_i - \hat{\gamma}_i\boldsymbol{\rho} + \frac{\delta\phi_i\hat{\gamma}_i}{1-\delta\omega}\boldsymbol{\rho}$.

## Proof of Proposition 6

Since the average surplus for each consumer type is the same in each period, we study the value $\bar{\bar{v}}(\delta) = \sum_{i\in\mathcal{I}}\hat{v}_i(\delta, \mathbf{q}_i(\delta), \mathbf{x}_i)$, where $\hat{v}_i(\delta, \mathbf{q}_i(\delta), \mathbf{x}_i) = \mathbf{q}_i^{*T}\big(\tilde{\mathbf{a}}_i - \frac{1}{2k_i}\tilde{\mathbf{B}}_i\mathbf{q}_i^*\big) - \big(\mathbf{q}_i^* - \mathbf{x}_i\big)^T\mathbf{p}^*(\delta)$. Separating the surplus of the consumer into the utility and price components, $\hat{v}_i^u(\delta, \mathbf{q}_i(\delta)) = \mathbf{q}_i^{*T}\big(\tilde{\mathbf{a}}_i - \frac{1}{2k_i}\tilde{\mathbf{B}}_i\mathbf{q}_i^*\big)$ and $\hat{v}_i^p(\delta, \mathbf{q}_i(\delta), \mathbf{x}_i) = \big(\mathbf{q}_i^* - \mathbf{x}_i\big)^T\mathbf{p}^*(\delta)$ the average aggregate surplus $\bar{\bar{v}}(\delta) = \sum_{i\in\mathcal{I}}\hat{v}_i(\delta, \mathbf{q}_i(\delta), \mathbf{x}_i) = \sum_{i\in\mathcal{I}}\hat{v}_i^u(\delta, \mathbf{q}_i(\delta)) + \sum_{i\in\mathcal{I}}\hat{v}_i^p(\delta, \mathbf{q}_i(\delta), \mathbf{x}_i) = \sum_{i\in\mathcal{I}}\hat{v}_i^u(\delta, \mathbf{q}_i(\delta))$, since the market clearing condition will lead to $\sum_{i\in\mathcal{I}}\hat{v}_i^p(\delta, \mathbf{q}_i(\delta) = 0$. The first derivative of aggregate surplus with respect to $\delta$ is

$$\frac{d}{d\delta}\bar{\bar{v}}(\delta) = \sum_{i\in\mathcal{I}}\frac{d}{d\delta}\hat{v}_i^u(\delta, \mathbf{q}_i(\delta)) = \sum_{i\in\mathcal{I}}\big(\nabla_{\mathbf{q}_i^*}\hat{v}_i^*(\delta, \mathbf{q}_i(\delta))\big)^T\frac{\partial}{\partial\delta}\mathbf{q}_i(\delta).$$

Recalling from the proof of Proposition 4 that $\tilde{\mathbf{a}}_i = \bar{\mathbf{B}}_i^{-1}\bar{\mathbf{a}}_i$ and $\hat{\gamma}_i\bar{\mathbf{B}}_i^{-1} = \bar{\bar{\mathbf{B}}}_i^{-1}$, the gradient of $\hat{v}_i$ with respect to $\mathbf{q}_i^*$ is

$$\begin{aligned}\nabla_{\mathbf{q}_i^*}\tilde{v}_i^*(\delta, \mathbf{q}_i^*, \mathbf{x}_i) &= \tilde{\mathbf{a}}_i - \bar{\mathbf{B}}_i^{-1}\mathbf{q}_i^*\\ &= \tilde{\mathbf{a}}_i - \bar{\mathbf{B}}_i^{-1}\left(\bar{\mathbf{a}}_i - \hat{\gamma}_i\boldsymbol{\rho} + \frac{\delta\phi_i\hat{\gamma}_i}{1-\delta\omega}\boldsymbol{\rho}\right)\\ &= \hat{\gamma}_i\bar{\mathbf{B}}_i^{-1}\boldsymbol{\rho} - \frac{\delta\phi_i\hat{\gamma}_i}{1-\delta\omega}\bar{\mathbf{B}}_i^{-1}\boldsymbol{\rho}\\ &= \left(1 - \frac{\delta\phi_i}{1-\delta\omega}\right)\bar{\bar{\mathbf{B}}}_i^{-1}\boldsymbol{\rho}\\ &= \left(\frac{1-\delta\big(\omega+\phi_i\big)}{1-\delta\omega}\right)\bar{\bar{\mathbf{B}}}_i^{-1}\boldsymbol{\rho}\\ &= \left(\frac{1-\delta\|\boldsymbol{\lambda}_i\|^2}{1-\delta\omega}\right)\bar{\bar{\mathbf{B}}}_i^{-1}\boldsymbol{\rho}.\end{aligned}$$

Since $\mathbf{q}_i^* = \bar{\mathbf{a}}_i - \hat{\gamma}_i \boldsymbol{\rho} + \frac{\delta \phi_i \hat{\gamma}_i}{1-\delta\omega}\boldsymbol{\rho}$, this implies that $\frac{\partial}{\partial \delta}\mathbf{q}_i(\delta) = \frac{\phi_i \hat{\gamma}_i}{\left(1-\delta\omega\right)^2}\boldsymbol{\rho}$ and that

$$\begin{aligned}\frac{d}{d\delta}\bar{\bar{v}}(\delta) &= \sum_{i\in\mathcal{I}} \left(\nabla_{\mathbf{q}_i^*}\tilde{v}_i^*(\delta, \mathbf{q}_i^*, \mathbf{x}_i)\right)^T \frac{\partial}{\partial \delta}\mathbf{q}_i(\delta) \\ &= \frac{\boldsymbol{\rho}^T \mathbf{B}^{-1}\boldsymbol{\rho}}{\left(1-\delta\omega\right)^3}\sum_{i\in\mathcal{I}}\left(1-\delta\|\boldsymbol{\lambda}_i\|^2\right)\phi_i\hat{\gamma}_i.\end{aligned}$$

Recall that $\sum_{i\in\mathcal{I}}\phi_i\hat{\gamma}_i = 0$ and so

$$\begin{aligned}\frac{d}{d\delta}\bar{\bar{v}}(\delta) &= -\frac{\boldsymbol{\rho}^T \mathbf{B}^{-1}\boldsymbol{\rho}}{\left(1-\delta\omega\right)^3}\delta\sum_{i\in\mathcal{I}}\|\boldsymbol{\lambda}_i\|^2\phi_i\hat{\gamma}_i \\ &= -\frac{\boldsymbol{\rho}^T \mathbf{B}^{-1}\boldsymbol{\rho}}{\left(1-\delta\omega\right)^3}\delta\left(\sum_{i\in\mathcal{I}}\hat{\gamma}_i\|\boldsymbol{\lambda}_i\|^4 - \omega^2\right).\end{aligned}$$

Indeed by the Cauchy-Schwarz inequality, which states that $(\sum_i x_i y_i)^2 \leq (\sum_i y_i)^2(\sum_i y_i)^2$, $\sum_{i\in\mathcal{I}}\hat{\gamma}_i\|\boldsymbol{\lambda}_i\|^4$ is always greater than or equal to $\omega^2$. To see this let $x_i = \sqrt{\hat{\gamma}_i}$ and $y_i = \sqrt{\hat{\gamma}_i}\|\boldsymbol{\lambda}_i\|^2$. This implies that $\omega_i^2 = (\sum_{i\in\mathcal{I}} x_i y_i)^2 \leq (\sum_{i\in\mathcal{I}} y_i)^2(\sum_{i\in\mathcal{I}} y_i)^2 = \sum_{i'\in\mathcal{I}}\hat{\gamma}_{i'}\sum\sum_{i\in\mathcal{I}}\hat{\gamma}_i\|\boldsymbol{\lambda}_i\|^4 = \sum_{i\in\mathcal{I}}\hat{\gamma}_i\|\boldsymbol{\lambda}_i\|^4$ and that $\frac{d}{d\delta}\bar{\bar{v}}(\delta)$ is nonnegative. Therefore, $\delta^* = 0$ is the level of strategic behavior that maximizes surplus.

# References


Adida, E. and DeMiguel, V. (2011). Supply chain competition with multiple manufacturers and retailers. *Operations Research*, 59(1):156–172.

Bapna, R., Jank, W., and Shmueli, G. (2008). Consumer surplus in online auctions. *Information Systems Research*, 19(4):400–416.

Bazhanov, A., Levin, Y., and Nediak, M. (2015). Quantity competition in the presence of strategic consumers. *Smith School of Business Working Paper*.

Besbes, O. and Zeevi, A. (2015). On the (surprising) sufficiency of linear models for dynamic pricing with demand learning. *Management Science*, 61(4):723–739.

Bhaskaran, S. R. and Gilbert, S. M. (2005). Selling and leasing strategies for durable goods with complementary products. *Management Science*, 51(8):1278–1290.

Boone, J. and Pottersz, J. (2006). Transparency and prices with imperfect substitutes. *Economics Letters*, 93(3):398–404.

Chesky, B. (2013). Who we are, what we stand for. *AirBnB Blog*.

Cohen, M. C., Perakis, G., and Pindyck, R. S. (2015). Pricing with limited knowledge of demand. *National Bureau of Economic Research Working Paper*.

Desai, P. and Purohit, D. (1998). Leasing and selling: Optimal marketing strategies for a durable goods firm. *Management Science*, 44(11-part-2):S19–S34.

Dobbs, R., Chen, Y., Orr, G., Manyika, J., Chang, E., and Institute, M. G. (2013). *China's E-tail Revolution: Online Shopping as a Catalyst for Growth*. McKinsey Global Institute.

Einav, L., Farronato, C., and Levin, J. (2015). Peer-to-peer markets. *Annual Review of Economics (Forthcoming)*.

Einav, L., Farronato, C., Levin, J. D., and Sundaresan, N. (2013). Sales mechanisms in online markets: What happened to internet auctions? Technical report, National Bureau of Economic Research.

Farahat, A. and Perakis, G. (2010). A nonnegative extension of the affine demand function and equilibrium analysis for multiproduct price competition. *Operations Research Letters*, 38(4):280–286.

Farahat, A. and Perakis, G. (2011). Technical note – a comparison of bertrand and cournot profits in oligopolies with differentiated products. *Operations Research*, 59(2):507–513.

Federgruen, A. and Hu, M. (2015). Multi-product price and assortment competition. *Operations Research*, 63(3):572–584.

Federgruen, A. and Hu, M. (2016). Technical note – sequential multiproduct price competition in supply chain networks. *Operations Research*, 64(1):135–149.

Goetzmann, W. N., Renneboog, L., and Spaenjers, C. (2011). Art and money. *The American Economic Review*, pages 222–226.

Gümüs, M., Ray, S., and Yin, S. (2013). Returns policies between channel partners for durable products. *Marketing Science*, 32(4):622–643.

Hendel, I. and Lizzeri, A. (1999). Interfering with secondary markets. *The Rand Journal of Economics*, pages 1–21.

Huang, S., Yang, Y., and Anderson, K. (2001). A theory of finitely durable goods monopoly with used-goods market and transaction costs. *Management Science*, 47(11):1515–1532.

Huet, E. (2015). How airbnb uses big data and machine learning to guide hosts to the perfect price. *Forbes*, pages http://www.forbes.com/sites/ellenhuet/2015/06/05/how–airbnb–uses–big–data–and–machine–learning–to–guide–hosts–to–the–perfect–price/.

Johnson, J. P. (2011). Secondary markets with changing preferences. *The RAND Journal of Economics*, 42(3):555–574.

Kachani, S. and Shmatov, K. (2011). Competitive pricing in a multi-product multi-attribute environment. *Production and Operations Management*, 20(5):668–680.

Kauffman, R. J., Spaulding, T. J., and Wood, C. A. (2009). Are online auction markets efficient? an empirical study of market liquidity and abnormal returns. *Decision Support Systems*, 48(1):3–13.

Kim, J., Allenby, G. M., and Rossi, P. E. (2002). Modeling consumer demand for variety. *Marketing Science*, 21(3):229–250.

Kluberg, J. and Perakis, G. (2012). Generalized quantity competition for multiple products and loss of efficiency. *Operations Research*, 60(2):335–350.

Liu, L., Shang, W., and Wu, S. (2007). Dynamic competitive newsvendors with service-sensitive demands. *Manufacturing & Service Operations Management*, 9(1):84–93.

Nagarajan, M. and Rajagopalan, S. (2009). Technical note – a multiperiod model of inventory competition. *Operations Research*, 57(3):785–790.

Olsen, T. L. and Parker, R. P. (2008). Inventory management under market size dynamics. *Management Science*, 54(10):1805–1821.

Olsen, T. L. and Parker, R. P. (2014). On markov equilibria in dynamic inventory competition. *Operations Research*, 62(2):332–344.

Oraiopoulos, N., Ferguson, M. E., and Toktay, L. B. (2012). Relicensing as a secondary market strategy. *Management Science*, 58(5):1022–1037.

Pang, J., Stone, R., and Cottle, R. (1992). The linear complementarity problem. *Computer Science and Scientific Computing. Academic Press*, 1:2.

Parker, J. A. and Vissing-Jorgensen, A. (2010). The increase in income cyclicality of high-income households and its relation to the rise in top income shares. Technical report, National Bureau of Economic Research.

Penasse, J. and Renneboog, L. (2014). Bubbles and trading frenzies: Evidence from the art market.

Renneboog, L. and Spaenjers, C. (2013). Buying beauty: On prices and returns in the art market. *Management Science*, 59(1):36–53.

Shen, Z.-J. M. and Su, X. (2007). Customer behavior modeling in revenue management and auctions: A review and new research opportunities. *Production and Operations Management*, 16(6):713–728.

Soon, W., Zhao, G., and Zhang, J. (2009). Complementarity demand functions and pricing models for multi-product markets. *European Journal of Applied Mathematics*, 20(5):399–430.

Sriram, S., Manchanda, P., Bravo, M. E., Chu, J., Ma, L., Song, M., Shriver, S., and Subramanian, U. (2013). Platforms: a multiplicity of research opportunities. *Marketing Letters*, 26(2):141–152.

Su, X. (2007). Intertemporal pricing with strategic customer behavior. *Management Science*, 53(5):726–741.

Yang, N. and Zhang, R. (2015). Dynamic competition under market size dynamics: Balancing the exploitation-induction tradeoff. *Olin Business School Working Paper*.

Yin, S., Ray, S., Gurnani, H., and Animesh, A. (2010). Durable products with multiple used goods markets: Product upgrade and retail pricing implications. *Marketing Science*, 29(3):540–560.