OPERATIONS RESEARCH
Vol. 62, No. 4, July–August 2014, pp. 846–863
ISSN 0030-364X (print) | ISSN 1526-5463 (online)

http://dx.doi.org/10.1287/opre.2014.1285

Quantity Premiums and Discounts in Dynamic Pricing

Yuri Levin, Mikhail Nediak, Andrei Bazhanov
School of Business, Queen's University, Kingston, Ontario, K7L 3N6, Canada
{ylevin@business.queensu.ca, mnediak@business.queensu.ca, andrei.bazhanov@queensu.ca}

We consider a dynamic pricing problem for a monopolistic company selling a perishable product when customer demand is both uncertain and occurs in batches that must be fulfilled as a whole. The seller can price-discriminate between different sized batches by setting different unit prices. The problem is modeled as a stochastic optimal control problem to find an inventory-contingent dynamic pricing policy that maximizes the expected total revenues. We find the optimal pricing policy and prove several monotonicity results. First, we establish stochastic order conditions on the unit willingness-to-pay distributions that determine when quantity discounts or premiums take place for a batch purchase compared to a rapid sequence of purchases with the same total size. Second, we give sufficient conditions for prices to be monotonically decreasing or increasing in inventory. Third, we characterize the conditions for the perceived quantity discounts and premiums that result from comparing unit prices for different batch sizes under a particular inventory level.

Subject classifications: dynamic pricing; batched demand; stochastic orders; dynamic programming applications.
Area of review: Transportation.
History: Received April 2012; revisions received January 2013, July 2013, January 2014, March 2014; accepted April 2014. Published online in *Articles in Advance* June 4, 2014.

1. Introduction

Dynamic pricing and other revenue management (RM) practices are becoming crucial in the operation of many industries including travel, hospitality, entertainment, and other services. To a great extent, this process is facilitated by the development of e-commerce that permits the sellers to dynamically adjust prices based on the current market environment and various inputs from potential buyers, including purchase size. In static pricing, charging different unit prices depending on the purchase size is referred to as nonlinear pricing. If the unit prices also vary with time, we talk about *dynamic nonlinear pricing* (DNP).

Customers are generally familiar with the practice of nonlinear pricing in retail. Examples include charging different unit prices for packages of different sizes and promotional strategies such as "buy two, get one free" or "buy one, get the second at 50% off." Bundling may also be viewed as a form of nonlinear pricing that involves multiple product types. However, in brick-and-mortar retail stores, pricing cannot be fully dynamic because the prices are typically fixed for a period of time. Customers can obtain inventory information by going to the store and, in their encounters with nonlinear prices, expect *quantity discounts*: a unit price that decreases with the purchase size. Indeed, an increasing unit price would simply result in customers making smaller, separate purchases. Implementing *quantity premiums* in retail requires limiting the number of purchases per customer within the time interval in which prices are fixed and, as a result, may alienate customers. Thus, quantity premiums in retail are almost never observed, except in situations that rely on the lack of customer attention to unit prices. Online retailers that rely on posted prices are also subject to customer expectations of quantity discounts. However, are such expectations always realistic in e-commerce systems where prices may vary among individual transactions? Perhaps quantity premiums can be more profitable for sellers in some nonretail situations?

Experience with online travel booking sites creates a perception that quantity premiums do occur. Repeated searches on Expedia.ca for a WestJet round-trip fare on a direct flight from Toronto to Cancun have revealed a startling 32% increase in unit prices as the number of seats required by the search increased from one to five (see Table 1). The table shows that there were several increases in unit price, indicating that WestJet employed a nonlinear pricing policy with quantity premiums. Such premiums are always present in traditional yield management systems with nested booking limits. Indeed, each subsequent seat only becomes more expensive for the customer as consecutive booking limits are getting filled. For a traditional system, perceptions of quantity premiums coincide with reality: if a customer attempts to break up a single large purchase into a rapid sequence of smaller ones with the same total size, the resulting total price remains the same. Additionally, if another customer "squeezes in" between the purchases in this sequence, the first customer's total price may become even higher or the required total number of seats may no longer be available.

Table 1. Price per ticket on WestJet roundtrip YYZ-CUN departing on September 17, 2011, and returning on September 24, 2011, and corresponding quantity premium as a percentage of a single ticket price.

No. of seats	Total price quoted ($)	Price/ticket (CAD) ($)	Percentage of premium
1	597.19	597.19	
2	1,254.38	627.19	5
3	1,881.57	627.19	5
4	2,668.76	667.19	12
5	3,935.95	787.19	32

Fixed prices (fare classes) used in traditional capacity-based yield management are generally somewhat suboptimal. In this paper, we study an optimal dynamic pricing model where prices are unconstrained and may potentially produce both quantity premiums and discounts. The model under consideration needs to take into account the limited time aspect and embed an element of knapsack-like packing of booking requests with different sizes into limited capacity. Markets where this study is especially important are characterized by customer segmentation according to the purchase size. The size-based market segments can then be represented by statistically independent arrival streams with their own intensities and (possibly different) distributions of customer willingness-to-pay (WTP). Although each customer may potentially require various amounts of the product, there are many contexts in which this amount is fixed before the customer makes a purchase decision. Examples include airline or hotel bookings where the size of the group is fixed, group purchases of event tickets, group admissions to recreational facilities (e.g., golf courses), and cargo shipments.

The assumption of statistically independent arrivals for each segment is standard in the pricing and revenue management theory (see, e.g., Talluri and van Ryzin 2004, §7.1). For size-based segments, this assumption precludes the choice of purchase size by an arriving customer; otherwise, the arrivals for each size would not be independent. Since the choice of purchase size is fixed externally, it is appropriate to call this an *exogenous demand* model. This model is an approximation of the real market, and the validity of this assumption depends on the effectiveness of barriers (also called "fences") that prevent customer migration between market segments. In most practical settings, for a batch of a given size, a customer can substitute a larger one if its price is lower but cannot substitute a single purchase of a smaller one. Therefore, batch size works as an effective fence only in the direction of smaller sizes. Also, if customers have good reasons to believe that they can benefit from breaking up a purchase into a series of smaller purchases, this may be enough of an incentive to "jump the fence" between the segments in the direction of smaller packages. A detailed analysis of practical ways to establish fences between size-based segments is beyond the scope of this work, but absent such fences, we can consider the properties of the optimal pricing policy based on exogenous demand that force customers to self-select into a correct segment.

In the context of the WestJet example, a quantity premium or discount is a monotonic property of the pricing policy. The goal of this paper is to describe the structure of the *optimal* pricing policy and study three classes of monotonic properties when demand intensity and customer WTP distribution for each market segment vary. First, monotonicity of prices as functions of the remaining inventory is a classical property considered in the literature. For example, Gallego and van Ryzin (1994) described the conditions under which the optimal price for one unit of a perishable product decreases in the remaining inventory.

The second type of monotonicity can only occur when we consider demand for batches of different size. We refer to this type as a *break-up monotonicity*, that is, a monotonic relation of the batch price relative to the total price realized were the batch to be broken up into a collection of subbatches purchased in rapid succession. This type of relation describes the actual quantity premiums and discounts that could be realized by the customers. Although customers cannot observe the actual quantity premium/discount unless they forgo the original batch purchase, i.e., replace it by a sequence of smaller purchases, the mere customer awareness of the qualitative properties of the policy can influence their behavior. If the batch price is higher than the sum of prices of subbatches, the pricing policy of the seller creates an incentive for customers to break up the batch. This incentive is reduced in the presence of customer transaction costs, or if customers have only limited information about prices at lower inventory levels and there is sizable supply and price risk. Thus, it may still be possible to implement quantity premiums under these conditions or when the seller has some other tools to prevent breakups (e.g., by tracking the identity of buyers). On the other hand, if the optimal batch price is lower than the sum of prices of subbatches, then an incentive to circumvent the system by breaking up the purchase is eliminated.

The third type of monotonicity is a relation between unit prices for batches of different size at the *same* inventory level (as opposed to break-up monotonicity that applies to purchases in rapid succession, i.e., at different inventory levels). It can be described as *perceived* quantity discounts or premiums. Understanding relations between unit prices may be important in appropriate presentation of prices to the customers and maintaining long-term customer perception of price fairness. One (in)famous example of a failure to properly frame price discounts/premiums is a statement by Coca Cola chairman and chief executive M. Douglas Ivester in connection to vending machines that can adjust prices depending on ambient temperature. As *New York Times* reported on October 28, 1999, he remarked in an interview to *Veja* that it is fair for a cold drink to be more expensive in summer heat, and the machine would make this process automatic. A better way to describe the same process would evidently be to say that the machine automates discounts in cold weather. The example presented in Table 1 represents the perceived quantity premium. The term "perceived" is appropriate in this case since we do not know how the price would actually behave had we purchased the tickets. Acting as customers, we would have to take both price and supply risks to determine this behavior. A simple explanation of reasons for perceived premiums may remove or at least reduce the potential concerns of customers about fairness.

In the analysis of the proposed model, we obtain the following insights. First, under a mild restriction that customer willingness-to-pay per batch is increasing (stochastically in the hazard rate order) with respect to the batch size, we establish that the optimal dynamic pricing policy results in prices that increase with batch size. Thus, a simple ordering of the WTP distributions effectively eliminates the need for explicit fences in the direction of larger batches.

Second, we use this characterization to analyze each of the three types of monotonic properties. Break-up monotonicity of prices is determined by the interaction of the order in WTP per unit of the batch and the convexity property of the reciprocal hazard rate of the WTP distribution. In particular, we prove that when customer WTP *per unit* of the batch is increasing (stochastically in the hazard rate order) with respect to batch size and the reciprocal of the hazard rate is concave, there is a quantity premium for the batch purchased as a whole compared to smaller batches purchased in rapid succession and totaling the same number of units. On the other hand, when WTP per unit of the batch is decreasing with respect to batch size and the reciprocal of the hazard rate is convex, there is a quantity discount (again relative to a breakup of the batch). As long as customers are aware of this property, an incentive to break up their batch purchases is eliminated. For the exponential WTP distribution (which has a constant hazard rate), this result provides a complete characterization of the break-up monotonicity property. Convexity (concavity) of the reciprocal hazard rate of WTP implies that the optimal price is convex (concave) in the unit cost for a general profit maximization problem corresponding to this WTP distribution. Surprisingly, the break-up monotonicity of prices is unrelated to the convexity properties of the problem's revenue-to-go (value) function, unlike the classical monotonicity that reduces to these properties, but is entirely determined by the properties of the WTP hazard rates.

Third, we consider monotonicity of prices with respect to the remaining inventory. This classical monotonicity is linked to the convexity properties of the value function of the problem (i.e., the optimal expected revenues given a particular remaining time and inventory level). Concavity (convexity) of the value function is also important from a managerial point of view because it represents opportunity costs, formally defined as first-differences of the value function in the level of inventory, that decrease (increase) in available capacity. Although neither concavity nor convexity of the value function can hold in our problem in general, we identify several settings where one of these properties is established conclusively. Near the end of the selling season, the sufficient condition for concavity (convexity) is that the expected optimal revenue rates for each batch size are strictly decreasing (increasing) in batch size. We also provide sufficient conditions on problem inputs that guarantee the concavity of the value function during the entire selling season. This result is stronger than the classical results concerning concavity of the value function in the absence of batch demand. In particular, it includes a finite upper bound on the difference between consecutive opportunity costs. The bound is expressed in terms of WTP distribution for a one-unit batch. We also find that, similarly to the dynamic pricing problem in the absence of batch demand, the concavity of the value function in inventory implies the concavity of the value function in time as well as the monotonic increasing property of opportunity costs and optimal prices in time.

The technical contribution of our results is the analysis of the continuous-time dynamics of opportunity costs based on a representation of optimal profits in terms of the hazard rate of the WTP distribution and the notion of the strengthened hazard rate order between unit and batch WTP distributions. The classes of distributions considered in the analysis are general and cover bounded as well as unbounded hazard rates. Similar approaches may be employed in other pricing problems.

Convexity and concavity of the value function also play a role in relations between prices for batches of different size at the same inventory level (perceived quantity discounts or premiums). For any pair of batch sizes, the relation between unit prices for different batch sizes is completely determined by the ratio of the unit opportunity costs and the ratio that measures the strength of the hazard rate order relation between customer WTPs *per unit* of the batch. As a consequence of this characterization, if the value function is concave in inventory and the customer unit WTP is increasing (stochastically in the hazard rate order) with respect to batch size, there are perceived quantity premiums. On the other hand, if the value function is convex and the customer WTP per unit is decreasing, there are perceived quantity discounts. Near the end of the planning horizon, the perceived quantity discount or premium is completely determined by the hazard rate order between unit WTP distributions for batches of different size. For the case of the maximum batch size of two, we identify conditions on the inputs that guarantee quantity discounts for sufficiently long time horizons. We apply numerical experimentation to further refine these insights.

The organization of the paper is as follows. In §2 we position our work within the existing literature. The model is presented in §3, and the main assumptions are introduced and results are preannounced in §4. The characterization of the optimal dynamic pricing policy and its monotonic properties are studied in §5. An illustration of main results for the case of exponential WTP distribution is presented in §6. Proofs of main text propositions/theorems and the statements of key technical lemmas are provided in the appendix. Proofs of all lemmas, examples of specific distributions satisfying our assumptions, and descriptions of numerical experiments can be found in the online appendix (available as supplemental material at http://dx.doi.org/10.1287/opre.2014.1285).

2. Literature

Nonlinear pricing is a widespread practice with areas of application including such important fields as international trade, telecommunications, transportation, energy, supply chains, and retail. Wilson (1993) provides an overview of the substantial economics and marketing literature on this subject. A classical approach to the problem in this literature is via the theory of incentives or mechanism design (see, for example, Laffont and Martimort 2001). This literature does not focus on the limited capacity or dynamic aspects of the problem that are critical to revenue management. Dhebar and Oren (1986) consider a dynamic model of pricing for a new product whose consumption value increases as the market expands (for example, telecommunications).

Starting with the classical work of Gallego and van Ryzin (1994), dynamic pricing of limited capacity over a finite horizon is one of the main approaches to RM (see Talluri and van Ryzin 2004, Chapter 5). Gallego and van Ryzin (1994) and subsequent works analyze the optimality conditions and the monotonic properties of the optimal pricing policy in various variants of this general setting. A typical assumption in this stream of literature is that customers demand one unit of the product at a time. One exception is Elmaghraby et al. (2008), who consider the design of optimal markdown mechanisms when rational customers have multiunit demands. However, in that model, the number of customers and their demands are known in advance, and there is no possibility to explicitly price-differentiate between customers who require different sizes. Dynamic nonlinear pricing can also be viewed as dynamic pricing for multiple products using a shared resource capacity. Such models were considered in the RM literature, for example, by Maglaras and Meissner (2006), who analyze a general model where each product uses one unit of a shared resource (unlike our case, in which the size variation between products is critical). This article presents a unified treatment of dynamic pricing and capacity control formulations, a reduction of the problem to the one based on the aggregate consumption rate, and asymptotically optimal heuristics based on the deterministic approximation. Some articles (for example, Zhang and Cooper 2009, Dong et al. 2009) consider dynamic pricing of substitutable perishable products. A distinguishing feature of these studies is that customers can choose between different products with independent resources. This is different from a setting (as in our paper) where the choice of the product is fixed in advance and the capacity is shared.

Nonlinear pricing is naturally embedded into the network RM models. In the general model of Gallego and van Ryzin (1997), demand requests of different size are represented as products with different capacity requirements. However, since the focus of that article is on the general network model, it does not discuss the monotonic structure of the optimal pricing policy that is required to answer the questions posed in our work. Gallego and van Ryzin (1997) also do not consider batch price order constraints that arise because a larger batch may be substituted for a smaller one. Generally, network RM literature is concerned with efficient computational methods for finding approximate control policies (see Talluri and van Ryzin 2004, Chapter 3), whereas we are interested in the structural properties of the optimal policy.

Batch demand has also been considered in the capacity-control and stochastic knapsack problems; see Lee and Hersh (1993) and van Slyke and Young (2000). Kleywegt and Papastavrou (2001), in the context of the stochastic knapsack problem, pointed out a potential nonconcavity of the value function and suggested sufficient condition for its concavity in terms of the conditional distribution function of the requested resource requirements given the associated reward.

To develop our analysis, we build on the technical approach employed by Gallego and van Ryzin (1994), Gallego and van Ryzin (1997) that views dynamic pricing as intensity control of a stochastic point process. To this, we add the analysis of problem inputs by means of stochastic order relations between customer willingness-to-pay random variables. For a review of stochastic orders, see Shaked and Shanthikumar (1994).

Customer perceptions of quantity premiums and discounts are related to the issues of price fairness. Such issues are studied in the consumer research literature using the notion of the reference price. In the setting of this paper, unit prices for different batch sizes provide possible reference price points. Examples of recent developments in this area include Bolton et al. (2003) and, in the context of dynamic pricing, Haws and Bearden (2006).

Finally, our model makes a common assumption that the inputs are known. However, this assumption is never satisfied exactly in practice, and there is a growing literature aimed at relaxing it. Recent studies of dynamic pricing in conjunction with demand learning include Araman and Caldentey (2009), Besbes and Zeevi (2009), and Farias and Van Roy (2010).

3. Model

We consider a problem of optimal dynamic pricing of a stock of $Y \geqslant 1$ identical perishable items over a time horizon of length $T > 0$ (using a reversed time index; i.e., T is the beginning and 0 is the end of the horizon). Customer demand is uncertain and occurs in batches that must be fulfilled as a whole. Thus, the seller can price-discriminate between the batches of different size by using different unit prices. Customer quote requests for batches of $i = 1, \ldots, \bar{y}$ items arrive according to statistically independent counting processes N_{is}, $0 \leqslant s \leqslant T$ with constant intensities λ_i. The quantity $\bar{y} \leqslant Y$ represents the maximum batch size. Upon request at time $s \in [0, T]$, the seller immediately quotes batch price p_{is} (using a nonanticipating pricing policy), and the purchase is completed by the customer with probability $\pi_i(p_{is})$. Resulting batch purchases of size i are described by "thinned" counting process $\tilde{N}_{is}$ whose intensity $\lambda_i \pi_i(p_{is})$ is price modulated.

3.1. Discussion of Problem Inputs: $\pi_i(p_i)$

Functions $\pi_i(p_i)$, $i = 1, \ldots, \bar{y}$ can be viewed as complementary cumulative distribution (survival) functions of given random variables as well as appropriately normalized aggregate demand functions. The latter are commonly assumed to be *regular* in the dynamic pricing literature (see Talluri and van Ryzin 2004, Assumption 7.1). Conditions (i)–(iv) of Assumption 7.1, in a probabilistic interpretation, can be expressed as follows:

Assumption 1 (Regularity I). *Each function $\pi_i(p_i)$, $i = 1, \ldots, \bar{y}$ is the complementary cumulative distribution (survival) function of the willingness-to-pay random variable W_i for a size i batch. W_i is an absolutely continuous nonnegative random variable with the density $f_i(p_i)$ whose support set is all nonnegative p_i.*

The condition on support sets makes this assumption somewhat stronger than Assumption 7.1. An immediate consequence is that the survival function has the inverse $p_i = \pi_i^{-1}(d_i)$, which is a strictly decreasing continuously differentiable function of purchase probability (demand) d_i (for all $d_i > 0$). Moreover, it is possible to express the expected batch revenues $\tilde{r}_i(p_i) = p_i \pi_i(p_i)$ per request as a function of d_i and the optimization problem in terms of d_i's as decision variables. Let $r_i(d_i) = \pi_i^{-1}(d_i) d_i$ denote the expected batch revenues as a function of batch purchase probability d_i. To forestall confusion, we use a superscript -1 next to the letter designating a function *only* to denote its inverse (as in $\pi_i^{-1}(d_i)$). This is different from the reciprocal value, which we denote by a superscript -1 next to the whole expression in round brackets (as in $(\pi_i(p_i))^{-1} = 1/\pi_i(p_i)$).

3.2. Dynamic Nonlinear Pricing as an Intensity Control Problem

The dynamic pricing problem formulation in this paper is closely related to the general dynamic pricing for network RM in Gallego and van Ryzin (1997), which is specialized here to a single resource and multiple products that require different amounts of this resource. The sets of feasible prices and purchase probabilities are denoted, respectively, by $\mathscr{P} = \{\mathbf{p} \in \mathbb{R}_+^{\bar{y}}\}$ and $\mathscr{D} = \{\mathbf{d} \in [0, 1]^{\bar{y}}\}$. For batches exceeding remaining inventory y, we use a standard convention that their prices are forced to ∞, effectively shutting down the demand since $\pi_i(p_i) \to 0$ as $p_i \to \infty$. Both this and an equivalent condition $d_i = 0$ for i in excess of the current capacity can be expressed by the constraint

$$\sum_{i=1}^{\bar{y}} \int_0^T i \, d\tilde{N}_{is} \leqslant Y \quad \text{(a.s.)}. \tag{1}$$

Let $\mathscr{U}$ be the set of all nonanticipating policies $\mathbf{p}_s \in \mathscr{P}$ (equivalently, $\mathbf{d}_s \in \mathscr{D}$), $s \in [0, T]$, which satisfy (1). The seller's problem is to find a control policy $u \in \mathscr{U}$ such that the expected total revenue

$$V^u(T, Y) = E_u\left[\sum_{i=1}^{\bar{y}} \int_0^T p_{is} \, d\tilde{N}_{is}\right] \tag{2}$$

is maximized. The optimal value, if it exists, is denoted as $V(T, Y) = \sup_{u \in \mathscr{U}} V^u(T, Y)$.

The stated problem is that of intensity control, with intensities expressed via purchase probabilities d_i as $\lambda_i d_i$. Sufficient optimality conditions to this problem are given by the Hamilton-Jacobi-Bellman equation (see Bremaud 1981, Theorem VII.1) for the revenue-to-go function $V(t, y)$ of remaining time $t \in [0, T]$ and inventory $y = 0, \ldots, Y$:

$$\frac{\partial}{\partial t} V(t,y) = \sup_{\mathbf{d} \in \mathscr{D}} \left\{ \sum_{i=1}^{y \wedge \bar{y}} \lambda_i [r_i(d_i) - d_i(V(t,y) - V(t,y-i))] \right\}, \quad \forall y = 1, \ldots, Y, \; t \in [0, T], \tag{3}$$

where we use a standard notational convention $y \wedge \bar{y} \triangleq \min\{y, \bar{y}\}$, with the boundary conditions

$$V(0, y) = 0, \quad \forall y = 1, \ldots, Y. \tag{4}$$

$$V(t, 0) = 0, \quad \forall t \in [0, T]. \tag{5}$$

Formally, the summation in Equation (3) excludes demand rates d_i for batch sizes i higher than remaining inventory y. This implicitly sets them to zero and ensures such demand is shut down.

The following condition (which implies Assumption 7.1(v)) is sufficient for the supremum to be attained in Equation (3) by a nontrivial maximizer:

Assumption 2 (Regularity II). $\lim_{d_i \to 0} r_i(d_i) = 0$, $i = 1, \ldots, \bar{y}$.

This condition implies that each $r_i(d_i)$ is continuous at 0 and, therefore, continuous on the entire interval [0, 1] (continuity for $d_i > 0$ follows from continuous differentiability of $\pi_i^{-1}(d_i)$). Continuity of each $r_i(d_i)$ implies continuity of the expression under the supremum in (3). This along with the observation that $\mathscr{D}$ is compact implies that the inputs to the problem satisfy conditions of Bremaud (1981, Theorem VII.3) that claims the existence of the unique solution to (3) and the corresponding Markovian optimal policy $d_i(t, y)$. (We use $p_i(t, y) = \pi_i^{-1}(d_i(t, y))$ to denote the corresponding optimal price.) However, the form of the optimal policy is fairly generic in this general setting and its properties are unclear. Thus, we next introduce additional assumptions on problem inputs. The main elements of notation used repeatedly throughout the paper are summarized in Table 2.

Table 2. Main notation.

Symbol	Definition
t, T	Remaining and initial length of time horizon
y, Y	Remaining and initial inventory
$\bar{y}$	Maximum batch size
λ_i	Arrival rate of demand requests for size i batches
W_i	WTP for size i batches
$\pi_i(p_i)$	Survival function of size i batch WTP
$f_i(p_i)$	Probability density function of size i batch WTP
$h_i(p_i)$	Hazard rate of size i batch WTP
$r_i(d_i)$	Expected revenue from size i batch arrival for purchase probability d_i
$J_i(d_i)$	Derivative of $r_i(d_i)$
$\tilde{J}_i(p_i)$	Reparametrization of $J_i(d_i)$ as a function of price p_i
$\rho_i(x)$	Optimal static profit at cost level x with demand function equal to $\pi_i(p)$
$\delta_i(x)$	Corresponding optimal static demand
$\xi_i(x)$	Corresponding optimal static price
$\xi_i(0)$, $\rho_i(0)$	Optimal revenue-maximizing static price and the corresponding optimal revenue
$V(t, y)$	Revenue-to-go for remaining time t and inventory y
$d_i(t, y)$	Corresponding optimal dynamic purchase probability for batch size i
$p_i(t, y)$	Corresponding optimal dynamic price
$\bar{p}_i(t, y)$	Corresponding optimal dynamic unit price
x_y	Opportunity cost of one item at inventory level y (a function of remaining time t)
$\dot{x}_y$	Its derivative with respect to t
$a(x_y)$	Upper bound on the difference $x_y - x_{y+1}$ for given x_y
$\bar{a}$	Global upper bound on $x_y - x_{y+1}$
$\bar{\gamma}$, γ	Power of lower and upper bounding polynomials for unbounded $h_i(p)$
$\bar{\epsilon}_i$	Strength of hazard rate order relation between W_i and W_{i+1}
ϵ_i	Strength of hazard rate order relation between W_i/i and $W_{i+1}/(i+1)$
β_i	Scaling parameter of exponential, Weibull, or gamma WTP distribution for batch size i

4. Main Assumptions and the Logical Structure of the Paper

As pointed out in §3, Assumptions 1 and 2 are standard regularity conditions on demand used in the pricing literature. The study of monotonic properties of the optimal pricing policy needs additional assumptions on monotonic properties of the model inputs: arrival rates and WTP distributions. Many of the results depend on the properties of hazard rates $h_i(p_i) \triangleq f_i(p_i)/\pi_i(p_i)$ of random variables W_i.

The first assumption discussed in §4.1 stipulates the increasing hazard rate property and facilitates the characterization of the optimal policy used in the rest of the paper. Many assumptions deal with *stochastic order* relations between WTP distributions. In §4.2, we take an opportunity to review two types of stochastic order relations: usual and hazard rate. This discussion clarifies why monotonicity results for prices require additional assumptions on the properties of hazard rates of WTP distributions. For example, the break-up monotonicity result of Theorem 1 in §5.1 relies on unit WTP that increases or decreases with respect to batch size in the hazard-rate order. The hazard rate stochastic order assumptions are also a key to generalizing classical monotonicity results to batches of the same size in §5.3 and analyzing perceived quantity premiums/discounts in §5.4. In §4.3, we strengthen hazard rate order relations by appropriately scaling unit and batch WTP random variables. We discuss how these strengthened relations are combined with restrictions on the request arrival rates λ_i and the growth of hazard rates.

4.1. An Assumption Used in the Characterization of the Optimal Policy

To guarantee that the first-order optimality condition is sufficient for the maximum in single-product pricing problems, it is frequently assumed that the revenue functions $r_i(d_i)$ are strictly concave; see, for example, Assumption 7.2 of Talluri and van Ryzin (2004). We make a somewhat stronger assumption superseding Assumption 2:

Assumption 3 (Increasing Hazard Rate). *For each $i = 1, \ldots, \bar{y}$, hazard rate $h_i(p_i)$ is increasing and continuously differentiable.*

This assumption is relatively easy to satisfy. Indeed, from representation $\ln \pi_i(p) = -\int_0^p h_i(\theta)\, d\theta$, it is immediate that an increasing hazard rate is equivalent to log-concavity of the survival function. The latter is true for any log-concave density, such as exponential, normal, gamma, Weibull, logistic and other distributions. To see that Assumption 3 guarantees strict concavity of $r_i(d_i)$'s, consider the marginal revenue rates

$$J_i(d_i) \triangleq r_i'(d_i) = \pi_i^{-1}(d_i) + d_i[\pi_i^{-1}(d_i)]' = \pi_i^{-1}(d_i) - \frac{d_i}{f_i(\pi_i^{-1}(d_i))} \tag{6}$$

and their reparametrization in terms of prices (obtained by substituting $d_i = \pi_i(p_i)$ in (6)):

$$\tilde{J}_i(p_i) \triangleq J_i(\pi_i(p_i)) = p_i - (h_i(p_i))^{-1} \tag{7}$$

(see Talluri and van Ryzin 2004, Equations (7.12) and (7.14)). The revenue function $r_i(d_i)$ is strictly concave if and only if $J_i(d_i)$ is strictly decreasing, and, in turn, if and only if $\tilde{J}_i(p_i)$ is strictly increasing. Increasing (not necessarily strictly) $h_i(p_i)$ is sufficient for the latter to hold.

Assumption 3 is used in Proposition 1, showing that the nonlinear dynamic pricing problem reduces to a collection of static univariate uncapacitated profit maximization problems corresponding to the customer population with WTP distribution W_i and cost $x \geqslant 0$. For each such problem, let $\rho_i(x)$ be its optimal value defined as

$$\rho_i(x) \triangleq \max_{d\in[0,1]} d\pi_i^{-1}(d) - xd, \tag{8}$$

and $\delta_i(x)$ be its optimal solution. Functions $\delta_i(x)$ and $\xi_i(x) \triangleq \pi_i^{-1}(\delta_i(x))$ of unit cost x represent the optimal static demand and price, respectively. When $x = 0$, problem (8) provides the static uncapacitated revenue-maximizing price $\xi_i(0)$ and the corresponding maximum static revenue $\rho_i(0)$ for demand function $\pi_i(p_i)$.

4.2. A Discussion of the Usual and the Hazard Rate Stochastic Orders

The *hazard rate order* $W_i \leqslant_{\text{hr}} W_{i+1}$ requires that the hazard rates satisfy inequality $h_i(p) \geqslant h_{i+1}(p)$ for all $p \geqslant 0$; see Shaked and Shanthikumar (1994) (further referred to as SS 1994). The *usual* stochastic order $W_i \leqslant_{\text{st}} W_{i+1}$ is weaker than the hazard rate order and is defined by the requirement on the survival functions: $\pi_i(p) \leqslant \pi_{i+1}(p)$ for all $p \geqslant 0$. The latter means that when offered a particular price p, the fraction of buying customers in the arrival stream for size $i+1$ batches is not less than the fraction of buying customers in the arrival stream for size i batches. Condition (1.B.3) in SS (1994) provides an alternative definition of the hazard rate order and immediately shows that it is stronger than the usual one. Either type of a stochastic order relation makes intuitive sense for our model since the fraction of customers who are prepared to pay a particular price for larger batches is usually larger.

Implications of stochastic orders for the market can be explained in terms of the graphs of purchase probabilities in logarithmic scale using representation $\ln \pi_i(p) = -\int_0^p h_i(\theta)\, d\theta$. Because of Assumption 3, both $\ln \pi_i(p)$ and $\ln \pi_{i+1}(p)$ are nonpositive concave decreasing functions equal to 0 for $p = 0$. Relation $W_i \leqslant_{\text{st}} W_{i+1}$ means that the graph of $\ln \pi_i(p)$ is below the graph of $\ln \pi_{i+1}(p)$. Relation $W_i \leqslant_{\text{hr}} W_{i+1}$ means that the absolute value of the slope of $\ln \pi_i(p)$ does not exceed the absolute value of the slope of $\ln \pi_{i+1}(p)$ for every p.

Stochastic order relations are useful in comparative statics results. For example, a larger WTP in hazard rate order leads to smaller marginal revenue rate (the proof is immediate):

Lemma 1. *Under Assumption 1 and condition $W_i \leqslant_{\text{hr}} W_{i'}$ for some $i < i'$, we have $\tilde{J}_i(p) \geqslant \tilde{J}_{i'}(p)$.*

In general, stronger order relations lead to stronger results in pricing models. For static pricing, the usual stochastic order relation leads to a comparative statics result for optimal profits, whereas the hazard rate order leads to a comparative statics result for optimal prices. To illustrate this fact in a general setting, we consider two WTP random variables W^A and W^B and assume that they satisfy Assumptions 1 and 3. Instead of subscript i, we endow all characteristics of these two distributions (such as survival functions and hazard rates) as well as characteristics of the optimal solutions to corresponding profit maximization problems (8) with superscripts A and B, respectively. Considering scaled versions of these WTP random variables, we obtain the following comparative statics:

Lemma 2. *Given $c^B, c^A > 0$, the usual stochastic order relation $W^B/c^B \leqslant_{\text{st}} W^A/c^A$ implies*
(a) $\rho^B(c^B x)/c^B \leqslant \rho^A(c^A x)/c^A$.
A hazard rate order relation $W^B/c^B \leqslant_{\text{hr}} W^A/c^A$ implies (a) *along with*
(b) $\xi^B(c^B x)/c^B \leqslant \xi^A(c^A x)/c^A$, *and*
(c) $c^A h^A(\xi^A(c^A x)) \leqslant c^B h^B(\xi^B(c^B x))$.

Additional insight into the structure of the optimal solution provided by the hazard rate stochastic order is clear. We use the scaling trick illustrated in this lemma to establish new structural results on the classical monotonicity properties and perceived quantity premiums and discounts.

4.3. Assumptions Used in Monotonicity Results

The policy characterization of Proposition 1 implies that batch prices increase with batch size if batch WTP increases with respect to batch size in the hazard rate order. The breakup monotonicity result of Theorem 1 uses the conditions that unit WTP decreases or increases with respect to batch size in the hazard rate order. In both cases, the conditions are relatively straightforward, and we defer their discussion until after the statements of these results.

Other monotonicity results require a more elaborate set of assumptions. There are two substantially different cases. The first, relatively simple case involves bounded hazard rates (such as the ones for exponential, gamma, or logistic distributions). However, for other common distributions (such as Weibull and normal), the hazard rate is unbounded. To handle that case, we make the following assumption:

Assumption 4 (Polynomial Growth of Hazard Rates). *There exists $0 \leqslant \bar{\gamma} \leqslant \gamma$ such that, for all $i = 1, \ldots, \bar{y}$, the hazard rate $h_i(x)$ satisfies the following condition:*

$$\left(\frac{\tilde{x}}{x}\right)^{\bar{\gamma}} h_i(x) \leqslant h_i(\tilde{x}) \leqslant \left(\frac{\tilde{x}}{x}\right)^{\gamma} h_i(x), \quad \textit{where } \tilde{x} \geqslant x \geqslant 0.$$

This assumption implies that the growth of the WTP hazard rate is bounded from above and below by the same power functions for all batch sizes. In terms of purchase probability, this means that the derivative of $\ln \pi_i(p)$ is bounded from above and below by a polynomial. This makes the rate of decrease in the purchase probability more consistent throughout the entire range of feasible prices and across different batch sizes. In general, the choice of constants $\bar{\gamma}$ and γ is not unique but should be chosen with γ as small and $\bar{\gamma}$ as large as possible to maximize the strength of this assumption for a given family of distributions.

The next set of assumptions tightens stochastic order relations between WTP for batches of consecutive sizes.

Assumption 5 (Tightened Hazard Rate Order). *For all $i = 1, \ldots, \bar{y}-1$ and some given $\epsilon_i, \bar{\epsilon}_i \geqslant 0$, WTP satisfies the following conditions:*

$$\frac{W_{i+1}}{i+1} \leqslant_{\text{hr}} \frac{W_i}{i(1+\epsilon_i)}, \quad \textit{and} \tag{9}$$

$$W_i \leqslant_{\text{hr}} \frac{W_{i+1}}{(1+\bar{\epsilon}_i)}. \tag{10}$$

Remark 1. Possible values of ϵ_i and $\bar{\epsilon}_i$ are restricted by inequality $(1+\epsilon_i)(1+\bar{\epsilon}_i) \leqslant (i+1)/i$. Indeed, the opposite inequality would make the pair of relations (9) and (10) infeasible. Moreover, since WTP hazard rates are increasing by Assumption 3, Assumption 5 with given $\bar{\epsilon}_i > 0$ or $\epsilon_i > 0$ implies it also holds with any smaller values of $\bar{\epsilon}_i$ or ϵ_i.

This suggests that quantity $(1+\epsilon_i)(1+\bar{\epsilon}_i)$ can be used as a measure of "strength" for Assumption 5.

If $\bar{\epsilon}_i = \epsilon_i = 0$, the requirements of this assumption take the form $W_{i+1}/(i+1) \leqslant_{\text{hr}} W_i/i$ and $W_i \leqslant_{\text{hr}} W_{i+1}$. The first condition states that WTP per unit is stochastically decreasing in batch size in the hazard rate order. The second condition states that WTP per batch is stochastically increasing in batch size in the hazard rate order. Both of these requirements are consistent with intuition since one would expect to pay more for a larger quantity of the product and simultaneously pay less per unit.

If $\bar{\epsilon}_i > 0$ or $\epsilon_i > 0$, the requirements become more stringent. The role of Assumption 5 in our analysis is twofold. First, in Assumption 6, it leads to significantly less restrictive conditions on the arrival rates λ_i than would be available otherwise. Second, we use it to identify conditions for perceived quantity premiums. Conditions (9) and (10) can be verified directly by examining the hazard rates of the given WTP distributions. Moreover, it is enough to ensure that they hold for the truncated version of the WTP random variables $W_i \mid W_i \geqslant \xi_i(0)$ since price values below the static uncapacitated revenue-maximizing price $\xi_i(0)$ cannot occur in the optimal solution.

For the exponential distribution, $h_i(x) = 1/\beta_i$ and Assumption 5 reduces to

$$1+\bar{\epsilon}_i \leqslant \frac{\beta_{i+1}}{\beta_i} \leqslant \frac{i+1}{i(1+\epsilon_i)}.$$

The choice of $\bar{\epsilon}_i$ and ϵ_i is not unique, but it makes sense to maximize the strength of the assumption by choosing $1+\bar{\epsilon}_i = \beta_{i+1}/\beta_i$ and $1+\epsilon_i = (i+1)\beta_i/i\beta_{i+1}$. As discussed next, this allows for the greatest flexibility in the arrival rates.

For the Weibull distribution, $h_i(x) = \zeta_i x^{\zeta_i - 1}/\beta_i$, Assumption 5 interacts with Assumption 4, resulting in combined restrictions on the parameters. In particular, Assumption 4 holds as long as $\bar{\gamma} \leqslant \zeta_i - 1 \leqslant \gamma$ for all i. When Assumptions 4 and 5 are combined, a stronger requirement of common $\zeta_i = \zeta$ results. Moreover, requirements of Assumption 5 take the form

$$(1+\bar{\epsilon}_i)^{\zeta} \leqslant \frac{\beta_{i+1}}{\beta_i} \leqslant \left(\frac{i+1}{i(1+\epsilon_i)}\right)^{\zeta}.$$

The last assumption identifies conditions for the arrival rates λ_i in terms of the properties of WTP distributions. It uses the related constants $\rho_i(0)$ and $\xi_i(0)$ representing the maximum expected revenue from a single booking request for a size i batch and the corresponding batch price (recall that $\rho_i(x)$ is defined by the uncapacitated static profit maximization problem (8) with unit cost x and $\xi_i(x)$ is the corresponding optimal price). We distinguish the *bounded* ($\sup_z h_i(z) < \infty$ for any $i \geqslant 1$) and *unbounded* ($\sup_z h_i(z) = \infty$ for any $i \geqslant 1$) hazard rate cases:

Assumption 6 (Bounds on Arrival Rates). *Arrival rates λ_i, $i = 1, \ldots, \bar{y}-1$ satisfy inequality*

$$\ln \frac{\lambda_{i+1}\rho_{i+1}(0)}{\lambda_i \rho_i(0)} + \Phi_i + \ln \max\{1, M_i\} < 0, \tag{11}$$

where Φ_i and M_i are constants defined in the bounded case by

$$M_i \triangleq \frac{\xi_i(0) \sup_z h_i(z)}{1+\bar{\epsilon}_i}, \tag{12}$$

$$\Phi_i \triangleq \frac{i+1}{1+\epsilon_i} \frac{1-i\epsilon_i}{2} \bar{a} h_{i+1}\left(\xi_{i+1}\left(\frac{(i+1)\bar{a}}{2\epsilon_i}\right)\right), \tag{13}$$

with $\bar{a} \triangleq \xi_1(0)\exp(1-[\xi_1(0)\sup_z h_1(z)]^{-1})$, and in the unbounded case by

$$M_i \triangleq \left(\frac{i+1}{i(1+\bar{\epsilon}_i)}\right)^{1+\gamma} \frac{i}{i+1}, \quad \text{and} \tag{14}$$

$$\Phi_i \triangleq \begin{cases} \left(\dfrac{\gamma}{\epsilon_i}\right)^{\gamma}\left(\dfrac{D_i(1+\epsilon_i)+\epsilon_i}{1+\gamma}\right)^{1+\gamma}, \\ \qquad\qquad \text{if } \epsilon_i \leqslant D_i[\gamma(1-i\epsilon_i)-(i+1)\epsilon_i], \\ [(i+1)D_i+1]^{\gamma}(1-i\epsilon_i)D_i, \quad \text{otherwise,} \end{cases} \tag{15}$$

with

$$D_i \triangleq [(1+\gamma)(1+\bar{\gamma})]^{1/(1+\bar{\gamma})} \frac{\xi_1(0)}{\xi_{i+1}(0)} \frac{1+\bar{\gamma}}{2+\bar{\gamma}} \cdot \frac{(i+1)^{(2+\bar{\gamma})/(1+\bar{\gamma})}-1}{i(1+\epsilon_i)}. \tag{16}$$

This assumption is an essential component of sufficient conditions for global concavity of the value function in the inventory level. All constants used in Assumption 6 are in terms of the inputs: WTP random variables W_i; constants directly derived from them ($\rho_i(0)$ and $\xi_i(0)$); constants $\bar{\gamma}$, γ, ϵ_i, and $\bar{\epsilon}_i$ used in Assumptions 4 and 5; and arrival rates λ_i. For arrival rates, condition (11) is simply an upper bound on the ratio of consecutive arrival rates λ_{i+1}/λ_i. This implies that for a given set of WTP distributions satisfying Assumptions 4 and 5, there always exist some sets of arrival rates that satisfy Assumption 6. Expressions $\lambda_{i+1}\rho_{i+1}(0)$ and $\lambda_i\rho_i(0)$ represent the maximum possible revenue rates per unit of time that can be generated from selling batches of size $i+1$ and i, respectively. With this interpretation, condition (11) is also an upper bound on the ratio of consecutive maximum revenue rates $(\lambda_{i+1}\rho_{i+1}(0))/(\lambda_i\rho_i(0))$.

After evaluating constants M_i and Φ_i for the exponential distribution, condition (11) takes the form

$$\ln \frac{\lambda_{i+1}\beta_{i+1}}{\lambda_i\beta_i} + \frac{i+1}{1+\epsilon_i}\frac{1-i\epsilon_i}{2}\frac{\beta_1}{\beta_{i+1}} < 0.$$

With the largest possible choice of $1+\epsilon_i = (i+1)\beta_i/i\beta_{i+1}$, the condition becomes

$$\ln \frac{\lambda_{i+1}\beta_{i+1}}{\lambda_i\beta_i} + \frac{i(i+1)\beta_1}{2}\left(\frac{1}{\beta_i} - \frac{1}{\beta_{i+1}}\right) < 0, \tag{17}$$

which is straightforward to check for any given inputs.

Detailed verifications of Assumptions 4–6 for Weibull, exponential, and gamma families of distributions are presented in the online appendix §EC.5.

5. Properties of the Optimal Policy

5.1. Characterization of the Optimal Pricing Policy

We now present a characterization of the optimal policy that serves as a foundation for the analysis of the monotonic properties. In particular, the solution to the univariate optimality conditions with respect to each d_i is not only feasible but also optimal for the multivariate problem in (3).

Proposition 1. *Under Assumptions* 1 *and* 3, *the pointwise optimization subproblem in* (3) *has a strictly concave objective function, and there exists the unique optimal policy determined by the univariate first-order optimality conditions; that is*

$$J_i(d_i(t,y)) = V(t,y) - V(t,y-i), \quad i = 1,\ldots,y\wedge\bar{y}. \tag{18}$$

The corresponding optimal prices are given by $p_i(t,y) = \pi_i^{-1}(d_i(t,y))$.

The managerial significance of Proposition 1 is that the optimal dynamic price *for each batch size* has the same property as in the single unit demand case; that is, the marginal revenue of the product is equal to the opportunity cost. This characterization also links decision variables for batches of different size:

Corollary 1. *For all* $t \geqslant 0$, $y \geqslant 2$, $i = 2,\ldots,y\wedge\bar{y}$, *and* $i_1 = 1,\ldots,i-1$ *policy variables satisfy*

$$\tilde{J}_i(p_i(t,y)) = \tilde{J}_{i_1}(p_{i_1}(t,y)) + \tilde{J}_{i-i_1}(p_{i-i_1}(t,y-i_1)). \tag{19}$$

The proof is immediate since $V(t,y) - V(t,y-i) = V(t,y) - V(t,y-i_1) + V(t,y-i_1) - V(t,y-i)$. In the next subsection, the additivity property (19) for marginal revenues corresponding to optimal batch prices helps to establish monotonicity relations between these prices.

Using the optimal static profit functions $\rho_i(x)$ defined by (8) and the corresponding optimal static demands $\delta_i(x)$, each term in the right-hand side of (3) can be represented as

$$r_i[\delta_i(V(t,y)-V(t,y-i))] - [V(t,y)-V(t,y-i)] \cdot \delta_i\big(V(t,y)-V(t,y-i)\big) = \rho_i\big(V(t,y)-V(t,y-i)\big).$$

Therefore, the summation and the maximization operators in (3) can be switched, and we obtain the optimality conditions in a more explicit form:

Corollary 2. *Equation* (3) *can be written as the ordinary differential equation*

$$\frac{\partial}{\partial t}V(t,y) = \sum_{i=1}^{y\wedge\bar{y}} \lambda_i \rho_i(V(t,y)-V(t,y-i)). \tag{20}$$

Term i in the summation represents the expected optimal revenue rate from size i batch when the opportunity cost is $V(t,y) - V(t,y-i)$.

Finally, Proposition 1 implies our first monotonicity result relating prices for batches of different size at each time instant and inventory level:

Corollary 3. *If* $W_i \leqslant_{\text{hr}} W_{i'}$ *for some* $1 \leqslant i < i' \leqslant \bar{y}$, *then* $p_i(t,y) \leqslant p_{i'}(t,y)$ *for all* $t \geqslant 0$, $i' \leqslant y \leqslant Y$.

This monotonicity result implies that batch prices are increasing in batch size whenever the input WTP distributions are such that $W_1 \leqslant_{\text{hr}} W_2 \leqslant_{\text{hr}} \cdots \leqslant_{\text{hr}} W_{\bar{y}}$. This addresses the first "practical" test for the pricing policy by preventing customers who require a batch of size i from substituting a batch of a larger size i'. Indeed, customers can "jump the fence" between the size segments risk-free and purchase a batch of a larger size than they actually need, if there is ever a violation of this increasing order, and it is free for customers to dispose the excess items. Free disposal applies to such items as tickets for group travel as long as a subgroup can still use a group ticket even if some members fail to show up.

For the rest of the paper, we require Assumptions 1 and 3.

5.2. Relations Between Prices of Batches and Those of Individual Items

We now apply Equation (19) k times breaking up the batch. Let the inventory levels considered in this breakup be $y - i \equiv y_{k+1} < y_k < \cdots < y_1 < y_0 \equiv y$ (where $i \leqslant \bar{y}$) and their differences be $i_j = y_j - y_{j+1}$, $j = 0,\ldots,k$. By Corollary 1, we obtain the relation

$$\tilde{J}_i(p_i(t,y)) = \sum_{j=0}^{k} \tilde{J}_{i_j}(p_{i_j}(t,y_j)).$$

Using (7), this relation can be further expanded to

$$p_i(t,y) - (h_i(p_i(t,y)))^{-1} = \sum_{j=0}^{k}\big(p_{i_j}(t,y_j) - (h_{i_j}(p_{i_j}(t,y_j)))^{-1}\big). \tag{21}$$

With $k = i-1$, the batch is broken up into individual items resulting in

$$p_i(t,y) - \big(h_i(p_i(t,y))\big)^{-1} = \sum_{j=0}^{i-1}\big(p_1(t,y-j) - \big(h_1(p_1(t,y-j))\big)^{-1}\big). \tag{22}$$

Compared to $p_i(t,y)$ (batch price), one can think of $p_{i_j}(t,y_j)$ as prices that would be paid by customers who purchased smaller batches of sizes i_j adding up to i in rapid succession. If the batch price $p_i(t,y)$ is above (below) the total price of subbatches $\sum_{j=0}^{k} p_{i_j}(t,y_j)$, then the optimal pricing policy exhibits actual quantity premiums (discounts). Note that such premiums or discounts are different from the perceived ones (discussed later in §5.4) resulting from the comparison of $p_i(t,y)/i$ and $p_{i_j}(t,y)/i_j$. The difference is that only the former can potentially be realized by a customer who needs to purchase a fixed quantity of i items. Equation (21) suggests that monotonicity relations between batch price $p_i(t,y)$ and the total price of subbatches $\sum_{j=0}^{k} p_{i_j}(t,y_j)$ do exist. These relations are described in the following proposition. We use the notion of the hazard rate order between *unit WTP* random variables $(1/i)W_i$, which have hazard rates $ih_i(iw)$ (here, w represents a value of WTP *per unit*). Therefore, $(1/i)W_i \leqslant_{\text{hr}} (\geqslant_{\text{hr}})(1/i')W_{i'}$, if and only if $ih_i(iw) \geqslant (\leqslant) i'h_1(i'w)$ for all w. An economic interpretation of $(1/i)W_i \leqslant_{\text{hr}} (\geqslant_{\text{hr}})(1/i')W_{i'}$ is that customers are generally prepared to pay less (more) per unit of size i batch than per unit of size i' batch.

Theorem 1 (Break-Up Monotonicity). *The following implications hold*:

(a) *If* $(1/i)W_i \leqslant_{\text{hr}} W_1$ *and* $(h_1(p_1))^{-1}$ *is convex, then* $p_i(t,y) \leqslant \sum_{j=0}^{i-1} p_1(t, y-j)$.

(b) *For* k, y_j, i_j *as defined above, if* $(1/i)W_i \leqslant_{\text{hr}} (1/i_j)W_{i_j}$ *for all* $j = 1, \ldots, k$ *and* $(h_i(p_i))^{-1}$ *is convex, then* $p_i(t,y) \leqslant \sum_{j=0}^{k} p_{i_j}(t, y_j)$.

(c) *If* $(1/i)W_i \geqslant_{\text{hr}} W_1$ *and* $(h_1(p_1))^{-1}$ *is concave, then* $p_i(t,y) \geqslant \sum_{j=0}^{i-1} p_1(t, y-j)$.

(d) *For* k, y_j, i_j *as defined above, if* $(1/i)W_i \geqslant_{\text{hr}} (1/i_j)W_{i_j}$ *for all* $j = 1, \ldots, k$ *and* $(h_i(p_i))^{-1}$ *is concave, then* $p_i(t,y) \geqslant \sum_{j=0}^{k} p_{i_j}(t, y_j)$.

Remark 2. The statement may be somewhat refined by an observation that $(h_1(p_1))^{-1}$ or $(h_i(p_i))^{-1}$ do not need to be concave or convex for all nonnegative prices. In particular, we only need concavity/convexity of $(h_1(p_1))^{-1}$ for $p_1 \geqslant \xi_1(0)$ since optimality conditions imply that $p_1(t,y) \geqslant \xi_1(0)$ for all t and y. Further refinement is possible if we consider upper bounds on prices that can arise within limited time horizons.

Parts (c) and (d) of Theorem 1 describe sufficient conditions for actual quantity premiums that can take place in practice if there exist fences that prevent customers from breaking up the batches. For example, event management applications for corporate customers and limited seating are conductive to quantity premiums when batch sales leave empty small blocks of seats that can be difficult to sell. Customer transaction costs, the lack of awareness about quantity premiums, or customer aversion to supply and price risks can also sustain break-up quantity premiums. On the other hand, if customers are aware of the properties of the optimal pricing policy, have negligible transaction costs, and can execute a very rapid sequence of purchases, parts (c) and (d) describe a setting in which the optimal policy provided by the model would not be sustainable.

Parts (a) and (b) describe the sufficient conditions under which the optimal policy exhibits actual quantity discounts. In this case, even if customers are aware of the properties of the optimal policy, they would have no incentive to break up the batch. If they break up the batch because of their lack of awareness, the policy generated by the model provides a lower bound on the revenue.

Theorem 1 uses the stochastic order conditions on unit WTP that are quite intuitive as well as conditions on convexity (or concavity) of the reciprocal of the hazard rate. This function is often called *Mill's ratio* in analysis and statistics; see, for example, Baricz (2008). The most direct implication of convexity (concavity) of $(h_i(p_i))^{-1}$ is that the marginal revenue rate $\tilde{J}_i(p_i)$ is concave (convex). This interpretation of convexity of the Mill's ratio was provided, for example, by Xu and Hopp (2009). Marginal revenue rate functions, along with the pricing theory, also arise in the mechanism design and auction theory, where they are usually referred to as virtual valuations; see Myerson (1981). Mierendorff (2011) uses convexity and concavity of the virtual valuations to determine whether strategic buyers truthfully reveal their time preferences (deadlines for purchase) to the mechanism designer. A further implication of convexity (concavity) of $(h_i(p_i))^{-1}$ is that the profit maximizing price $\xi_i(x)$ corresponding to WTP for size i batch is convex (respectively, concave) in opportunity cost x. This observation is a key to understanding the role of convexity and concavity of $(h_i(p_i))^{-1}$ in Theorem 1. Indeed, since the opportunity cost associated with selling a batch can be additively decomposed into opportunity costs associated with selling several smaller batches of the same total size, convex (concave) $\xi_i(x)$ tends to decrease (increase) the batch price relative to the sum of prices of smaller batches. We illustrate conditions of Theorem 1 on several examples.

A ready illustration for Theorem 1 is the case of exponential WTP distribution that has a constant hazard rate (consequently, the reciprocal hazard rate is increasing and convex as well as concave). The hazard rate $h_i(p_i)$ is equal to $1/\beta_i$, where β_i is the mean of W_i. Condition $(1/i)W_i \leqslant_{\text{hr}} (\geqslant_{\text{hr}}) W_1$ is equivalent to $\beta_i \leqslant (\geqslant) i\beta_1$. From of Theorem 1(a) and 1(c), we have $p_i(t,y) \leqslant (\geqslant) \sum_{j=0}^{i-1} p_1(t, y-j)$ if and only if $\beta_i \leqslant (\geqslant) i\beta_1$. Thus, the optimal pricing policy will result in a quantity premium (compared to individual items purchased in rapid succession) when $\beta_i \geqslant i\beta_1$ and in a quantity discount when $\beta_i \leqslant i\beta_1$. Effectively, Theorem 1 provides necessary and sufficient conditions for the case of exponential WTP distribution.

An exponential distribution may be about the only one with the globally concave reciprocal hazard rate, increasing hazard rate, and the support set of all nonnegative numbers. On the other hand, there are several widely used distributions for which the reciprocal of the hazard rate is convex. Examples include logistic, normal, gamma, and Weibull distributions; see Xu and Hopp (2009). A combination of such common distributions with stochastically decreasing $(1/i)W_i$ (in the hazard rate order) guarantees a discount when buying a batch as a whole compared to breaking up this batch into a rapid sequence of smaller purchases.

Although conditions of parts (a) and (b) appear as more likely to hold in practice, we point out that they are generally not "only if" (necessary) conditions. In other words, even though conditions of parts (c) and (d) do not hold, it is still possible that (a) and (b) do not hold either and the optimal batch price exhibits a premium. When reciprocal hazard rates are convex, the theorem shows that premiums are possible only if unit WTP is stochastically increasing in batch size.

5.3. Monotonic Properties of Prices for Batches of the Same Size

The characterization of the optimal policy provided by Proposition 1 includes the optimality condition for size-one batch as a special case: $J_1(d_1(t,y)) = V(t,y) - V(t,y-1)$. Since $J_1(d_1)$ is decreasing in d_1, $d_1(t,y)$ is increasing in y if and only if $x_y \triangleq V(t,y) - V(t,y-1)$ is decreasing in y. The latter condition, which describes a decreasing opportunity cost x_y associated with selling one unit of capacity, is equivalent to the concavity of $V(t,y)$—the property established by Gallego and van Ryzin (1994) for a single-unit dynamic pricing model. It is also immediate to see that concavity of $V(t,y)$ implies that $d_i(t,y)$ are increasing in y for each i. However, is $V(t,y)$ concave in general? Is it possible that $V(t,y)$ is convex under some conditions?

The following proposition provides conditions under which concavity and convexity hold near the end of the planning horizon. Along with other results of this section, the proposition is established by the analysis of differential equations describing the dynamics of the opportunity costs:

$$\dot{x}_y = \sum_{i=1}^{y\wedge\bar{y}} \lambda_i \rho_i (x_y + \cdots + x_{y-i+1}) - \sum_{i=1}^{(y-1)\wedge\bar{y}} \lambda_i \rho_i (x_{y-1} + \cdots + x_{y-i}), \quad y = 1, \ldots, Y \tag{23}$$

with the boundary conditions

$$x_y(0) = 0, \quad y = 1, \ldots, Y.$$

Proposition 2 (Local Concavity and Convexity near $t = 0$). *For any y such that*

$$\lambda_y \rho_y(0) > \lambda_{y+1} \rho_{y+1}(0) \quad or \tag{24}$$

$$\lambda_y \rho_y(0) < \lambda_{y+1} \rho_{y+1}(0) \tag{25}$$

holds, we have, respectively, $V(t,y) - V(t,y-1) > (<) V(t,y+1) - V(t,y)$ for all sufficiently small $t > 0$. Moreover, if (24) holds for all $y = 1, \ldots, \bar{y} - 1$, then there exists $\bar{t} > 0$ such that $V(t,y)$ is strictly concave in $y = 1, \ldots, Y$ for all $t \in (0, \bar{t})$. Also, if $\bar{y} = Y$ and (25) holds for all $y = 1, \ldots, Y-1$ then there exists $\bar{t}$ such that $V(t,y)$ is strictly convex in $y = 1, \ldots, Y$ for all $t \in (0, \bar{t})$.

Conditions used in this proposition require that under zero opportunity costs the expected optimal intensity of revenue per unit of time for batch size y is decreasing (increasing) in y. In general terms, we can conclude that the opportunity costs are decreasing (increasing) in inventory if the amounts of revenue derived from larger batches are smaller (larger). Indeed, if the intensity of revenue for size $y+1$ batch is larger than for size y batch, then the $y+1$th item is more valuable than yth item, pushing the value function at $y+1$ toward convexity. Condition $\lambda_y \rho_y(0) > \lambda_{y+1} \rho_{y+1}(0)$, which is equivalent to $\ln(\lambda_{y+1} \rho_{y+1}(0))/(\lambda_y \rho_y(0)) < 0$, is a weakened version of condition (11) of Assumption 6, whereas its opposite $\lambda_y \rho_y(0) < \lambda_{y+1} \rho_{y+1}(0)$ indicates a violation of (11). This observation clarifies the role of Assumption 6 in our global concavity result below.

Conditions (24) and (25) can be easily verified from problem inputs. Computing the maximum revenue per batch $\rho_i(0)$ requires solving the first-order optimality condition $\tilde{J}_i(\xi_i(0)) = 0$ for $\xi_i(0)$ (recall that $\tilde{J}_i(p_i)$ is a reparametrization of the marginal revenue rate in terms of price) and evaluating $\rho_i(0) = \pi_i(\xi_i(0))\xi_i(0)$. If W_i is exponential with mean β_i, conditions (24) and (25) reduce, respectively, to $\lambda_y \beta_y > \lambda_{y+1} \beta_{y+1}$ and $\lambda_y \beta_y < \lambda_{y+1} \beta_{y+1}$ (since $\rho_i(x) = \beta_i \exp(-1 - x/\beta_i)$). In general, we can infer (24) and (25) from the following stochastic order properties of WTP random variables:

Lemma 3. *If there exists $\Delta > 0$ such that $\lambda_y W_y \geqslant_{\text{st}} (1+\Delta) \cdot \lambda_{y+1} W_{y+1}$ (respectively, $(1+\Delta)\lambda_y W_y \leqslant_{\text{st}} \lambda_{y+1} W_{y+1}$), then condition (24) (respectively, (25)) holds.*

This claim is quite intuitive and can be established by the direct application of Lemma 2 (a). It shows that the usual stochastic order relations between scaled WTP distributions imply *local* monotonicity results for the revenue-to-go function and optimal policy. Proposition 2 and Lemma 3 also make the point that in the presence of batch demand, the concavity of the value function is a result of joint restrictions on the arrival rates *and* on the WTP distributions, even near the end of the planning horizon. On the other hand, *global* concavity of the value function (decreasing opportunity costs) in inventory may be expected from economic intuition and has important implications for the dynamic properties of the value function and optimal control policies. This guides further study of concavity starting with the following:

Theorem 2 (Monotonicity in t). *If $V(t,i)$ is (strictly) concave in i for all $i = 1, \ldots, y$ and $t > 0$, then, for all $t \geqslant 0$,*

(a) *$V(t,y)$ is (strictly) concave in t,*
(b) *x_y is (strictly) increasing in t,*
(c) *$d_i(t,y), i = 1, \ldots, y \wedge \bar{y}$ is (strictly) decreasing in t, and*
(d) *$p_i(t,y), i = 1, \ldots, y \wedge \bar{y}$ is (strictly) increasing in t.*

In practice, the strict concavity of the value function in time guarantees the unique solution to the problem of the optimal selection of the length of the selling horizon with a constant cost of running the booking system per unit of time.

Theorem 2 directly generalizes analogous statement of Gallego and van Ryzin (1994) without requiring new assumptions for its proof. It is then quite surprising that the proof of concavity of $V(t,y)$ in y requires a completely new result about the behavior of difference $x_{y-1} - x_y$ between the consecutive opportunity costs as a function of time. It is even more surprising that this result cannot be established outside the inductive proof of $x_y \leqslant x_{y-1}$ unless the maximum batch size $\bar{y}$ is at most two. The form of the upper bound on the difference between the opportunity costs is

$$x_{y-1} - x_y \leqslant a(x_{y-1}) \triangleq \rho_1(x_{y-1}) \int_0^{x_{y-1}} \frac{d\theta}{\rho_1(\theta)}. \qquad (26)$$

To the best of our knowledge, inequality (26) has not been established in the prior literature even in the absence of batch demand.

The following monotonic properties in inventory level global (hold for all times and inventory levels) and rely on hazard rate order relations between scaled WTP random variables.

Theorem 3 (Monotonicity in y). *Suppose Assumptions 4–6 hold, where Assumption 5 holds with $\epsilon_i > 0$ in the unbounded hazard rates case. Then for all $t \geqslant 0$,*

(a) $x_{y+1} \leqslant x_y$ *(i.e., $V(t, y)$ is concave in y), $y = 1, \ldots, Y-1$,*

(b) $x_{y+1} \geqslant x_y - a(x_y)$, $y = 1, \ldots, Y-1$, *where $a(x_y)$ are uniformly bounded from above by $\bar{a}$, $x_y \to \infty$ and, if $h_1(x)$ is unbounded, $a(x_y) \to 0$ as $t \to \infty$,*

(c) $d_i(t, y), i = 1, \ldots, y \wedge \bar{y}$ *is increasing in* $y = 1, \ldots, Y$, *and*

(d) $p_i(t, y), i = 1, \ldots, y \wedge \bar{y}$ *is decreasing in* $y = 1, \ldots, Y$. *Parts* (a), (c), *and* (d) *hold in the strict sense for* $t > 0$.

Parts (a), (c), and (d) form the classical set of monotonic properties of the value function and the optimal policy in the level of remaining inventory y. In combination with Theorem 2, the above statement ensures that the optimal pricing policy has intuitive properties: prices increase in the amount of time available and decrease in the available inventory. Similarly to Theorem 2, the strict concavity of the value function in inventory guarantees the unique solution to the problem of selecting the initial capacity Y under constant unit costs. Moreover, as we see in the next subsection, decreasing opportunity costs provide a possible explanation for perceived quantity premiums by suggesting that remaining items become more valuable as sales progress.

Part (b) of the theorem asserts unbounded property of the opportunity costs while their differences are bounded. We recall from the statement of Assumption 6 that $\bar{a} = \xi_1(0)\exp(1 - [\xi_1(0)\sup_z h_1(z)]^{-1})$, which is equal to β_1 for the exponential distribution. Opportunity costs are unbounded because the support sets of WTP distributions are unbounded, and for every unit in inventory, the number of selling opportunities becomes essentially infinite as the time horizon increases. When WTP hazard rates are unbounded, part (b) asserts that the value function becomes more and more "linear" as $t \to \infty$.

5.4. Relations Between Unit Prices for Batches of Different Size

Recall that break-up monotonicity properties discussed in §5.2 depend only on the characteristics of the hazard rates for unit WTP $(1/i)W_i$. On the other hand, same-size monotonicity properties depend on the concavity properties of the value function $V(t, y)$ with respect to y. Relations between unit prices discussed in this section depend on both. We use the following characterization of concavity: $V(t, y)$ is concave (convex) in y if and only if

$$\frac{V(t, y) - V(t, y-i)}{i} \leqslant (\geqslant) \frac{V(t, y) - V(t, y-i')}{i'}$$

for all $0 < i < i' \leqslant y$. Ratio $(V(t, y) - V(t, y-i))/i = (x_y + \cdots + x_{y-i+1})/i$ is the *unit* opportunity cost when selling the batch of size i. We also introduce the optimal unit price for the batch of size i as $\bar{p}_i(t, y) \triangleq p_i(t, y)/i$. The relation between unit prices becomes transparent if we restate the optimal policy characterization (18) as

$$\bar{p}_i(t, y) - (ih_i(i\bar{p}_i(t, y)))^{-1} = \frac{V(t, y) - V(t, y-i)}{i}. \qquad (27)$$

We can measure the magnitude or strength of the order relation between unit opportunity costs at time t and inventory level y by the value $P_{ii'}(t, y)$ such that

$$P_{ii'}(t, y)\frac{V(t, y) - V(t, y-i)}{i} = \frac{V(t, y) - V(t, y-i')}{i'}.$$

The value $P_{ii'}(t, y)$ is well defined for any $t > 0$ such that unit opportunity cost $(V(t, y) - V(t, y-i))/i$ is not zero. If $P_{ii'}(t, y) > 1$ ($P_{ii'}(t, y) < 1$), then the unit opportunity costs are related in the direction of concavity (convexity). Similarly, we can measure the magnitude of the stochastic order relation between $(1/i)W_i$ and $(1/i')W_{i'}$. Recall that $(1/i)W_i \leqslant_{\text{hr}} (\geqslant_{\text{hr}}) (1/i')W_{i'}$ if $ih_i(iw) \geqslant (\leqslant) i'h_{i'}(i'w)$ for all $w \geqslant 0$. The latter is equivalent to

$$\frac{J_i(iw)}{i} = w - (ih_i(iw))^{-1} \geqslant (\leqslant) w - (i'h_{i'}(i'w))^{-1} = \frac{J_{i'}(i'w)}{i'}, \quad \text{for all } w \geqslant 0.$$

We are particularly interested in measuring the magnitude of the order relation at the optimal unit price $w = \bar{p}_i(t, y)$ and, therefore, define this magnitude as $Q_{ii'}(t, y)$ such that

$$Q_{ii'}(t, y)\left(\bar{p}_i(t, y) - \left(ih_i(i\bar{p}_i(t, y))\right)^{-1}\right) = \bar{p}_i(t, y) - \left(i'h_{i'}(i'\bar{p}_i(t, y))\right)^{-1}.$$

If $(1/i)W_i \leqslant_{\text{hr}} (\geqslant_{\text{hr}}) (1/i')W_{i'}$, then $Q_{ii'}(t, y) \leqslant (\geqslant) 1$. We have the following characterization:

Lemma 4. *$\bar{p}_i(t, y) \leqslant \bar{p}_{i'}(t, y)$ if and only if $Q_{ii'}(t, y) \leqslant P_{ii'}(t, y)$.*

This statement provides a comparison between two factors influencing unit prices: the unit opportunity cost and the unit WTP. A larger batch of size i' has a higher unit opportunity cost (described by $P_{ii'}(t, y) > 1$) if the value function exhibits concave behavior. This pushes the unit price for a size i' batch upward compared to size i. On the other hand, the unit price is also limited by how much customers are willing to pay per unit of a size i' batch. When they are willing to pay more than per unit of a smaller size i batch, we have $Q_{ii'}(t,y) \leqslant 1$. In this case, the optimal pricing policy is guaranteed to exhibit the perceived quantity premium. On the other hand, the perceived quantity premium for larger batches may still be realized even if their unit WTP is smaller ($Q_{ii'}(t,y) \geqslant 1$). This occurs when $1 \leqslant Q_{ii'}(t,y) < P_{ii'}(t,y)$, i.e., as long as batch i' unit WTP is not too small. Similarly, when the size i' unit WTP is larger ($Q_{ii'}(t,y) \leqslant 1$), it is still possible that the lower unit opportunity cost (a convex value function or $P_{ii'}(t,y) < 1$) will cause quantity discounts. This occurs when $P_{ii'}(t,y) < Q_{ii'}(t,y) \leqslant 1$.

The above discussion of Lemma 4 shows that the order between unit prices at the same inventory level (perceived quantity discounts or premiums) is determined by the relative strength of the stochastic order relations between unit WTP and the strength of concavity (convexity) of the value function. Based on this discussion, we can point to two cases where the order between unit prices is particularly clear:

Corollary 4. *If $V(t,y)$ is concave in y and, for some $i < i'$, $(1/i)W_i \leqslant_{\text{hr}} (1/i')W_{i'}$, then $\bar{p}_i(t,y) \leqslant \bar{p}_{i'}(t,y)$. On the other hand, if $V(t,y)$ is convex in y and, for some $i < i'$, $(1/i')W_{i'} \leqslant_{\text{hr}} (1/i)W_i$, then $\bar{p}_{i'}(t,y) \leqslant \bar{p}_i(t,y)$.*

This corollary provides sufficient conditions for order relations between unit prices. The first part says that if opportunity costs are decreasing (that is, the value function is concave) and customers in different size segments are generally willing to pay more per unit as the batch size increases, then unit prices are increasing in batch size. By Proposition 2, concavity of the value function near the end of the planning horizon requires only the ordering (24) of the maximum revenue rates. Thus, this scenario appears quite plausible, e.g., in event management. On the other hand, if opportunity costs are increasing and customers are willing to pay less as the batch size increases, then unit prices are decreasing in batch size.

Lemma 4 points to relations between unit opportunity costs and unit WTP distributions as the main drivers of the ordering between unit prices. However, it is not clear which of these two drivers prevails in any given situation (in terms of problem inputs, current inventory level y, and time t).

The order between unit prices can be clearly established based on the order relation between unit WTP distributions near the end of the planning horizon (opportunity costs are near zero). We establish a more general result:

Proposition 3 (Local Perceived Quantity Premiums and Discounts). *If there exists $\Delta > 0$ such that $(1+\Delta)\cdot(W_k/k) \leqslant_{\text{hr}} W_{k'}/k'$ and the unit opportunity costs for the batches of sizes k and k' are equal at a particular time t_0 and inventory level y, i.e., $(V(t_0,y)-V(t_0,y-k))/k = (V(t_0,y)-V(t_0,y-k'))/k'$, then $\bar{p}_k(t,y) < \bar{p}_{k'}(t,y)$ in some neighborhood of t_0.*

The role of Δ in this proposition is to ensure that the order relation is in a sense "strict." However, a strict stochastic order relation is not a classical notion. Generally, a larger value of Δ results in a larger interval around t_0 where the result of the proposition holds.

Since opportunity costs at time $t = 0$ are zero, the proposition implies that, near the end of the planning horizon, we have $\bar{p}_k(t,y) < \bar{p}_{k'}(t,y)$ regardless of the inventory level. Further from the end of the planning horizon, the analysis of the relations between unit prices becomes difficult because of a complex interplay between concavity/convexity properties of $V(t,y)$ and stochastic order properties of the WTP distributions. For large t, under assumptions that guarantee concavity of the value function, it is possible to establish the following (quantity discount) relation between unit prices for 1 and 2 items.

Lemma 5. *If the conditions of Theorem 3 hold as well as in Assumption 4, we have $\gamma \in [0, 0.5(\sqrt{5}-1)]$, and under Assumption 5 $\epsilon_1 \geqslant \ln 2/(2-\ln 2) = 0.5304$, then there exists such $\bar{t} > 0$ that for any $t \geqslant \bar{t}$,*

$$\bar{p}_1(t,y) \geqslant \bar{p}_2(t,y). \tag{28}$$

Although the concavity of the value function generally favors quantity premiums, this lemma shows that a sufficiently strong stochastic order for unit WTP favoring discounts prevails sufficiently far from the end of the planning horizon. Global results require a more refined bound on the difference between opportunity costs than the one established in Theorem 3. In the next section, we present such a bound for the case of exponential WTP distributions.

6. Summary of Insights for the Case of Exponential Distribution

In this section, we summarize all of the obtained structural results for the case of exponential distribution. First, batch prices are increasing in batch size by Corollary 3 if the means of batch WTP distributions are restricted by $\beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_{\bar{y}}$.

Second, we recall that conditions of Theorem 1 provide a complete characterization of break-up monotonicity since $h_i(p)$ is constant for the exponential distribution. The optimal policy will result in quantity discounts when buying in batches relative to single units in rapid succession if and only if $\beta_i/i \leqslant \beta_1$ (parts (a) and (c)).

Third, by Proposition 2, concavity of the revenue-to-go function $V(t,y)$ in $y = 1,\ldots,Y$ (convexity in $y = 1,\ldots,\bar{y}$) near $t = 0$ holds if and only if $\lambda_{i+1}\beta_{i+1} < (>)\lambda_i\beta_i$ for all $i = 1,\ldots,\bar{y}-1$. By Theorem 3, concavity of $V(t,y)$ in y holds for all t if $\beta_i \leqslant \beta_{i+1}$, $\beta_{i+1}/(i+1) \leqslant \beta_i/i$ and (17) holds for all $i = 1,\ldots,\bar{y}-1$. Part (d) shows the monotonic decreasing property of the optimal prices in y, and Theorem 2 shows concavity of the revenue-to-go function as well as the monotonic increasing property of the optimal prices in t. At the same time, part (b) provides the upper bound β_1 on the difference in any pair of consecutive opportunity costs.

Fourth, Corollary 4 applies when both the value function is concave (convex) and $\beta_i/i \leqslant (\geqslant)\beta_{i'}/i'$ for some $i < i'$. Concavity (convexity) holds regardless of the latter condition, for example, near $t = 0$ when $\lambda_{i+1}\beta_{i+1} < (>)\lambda_i\beta_i$ for all $i = 1, \ldots, \bar{y}-1$. The implication of Corollary 4 under these conditions is that $\bar{p}_i(t, y) \leqslant (\geqslant)\bar{p}_{i'}(t, y)$. By Proposition 3, strict condition $\beta_i/i < (>)\beta_{i'}/i'$ for $i < i'$ also guarantees $\bar{p}_i(t, y) < (>)\bar{p}_{i'}(t, y)$ in the neighborhood of any point t_0 at which the unit opportunity costs for batches of sizes i and i' are equal.

Finally, for the exponential distribution, there is a global (for all t and y) result that is stronger than the asymptotic result of Lemma 5. This result includes a global refinement on the bound (26).

Lemma 6. *Under the conditions of Theorem* 3, *the exponential case* ($\rho_1(x) = \beta_1 e^{-1-x/\beta_1}$) *yields, for any* $t > 0$, *the bound*

$$a_{y-1} \triangleq x_{y-1} - x_y \leqslant \beta_1 \ln\left[2\left(1+\exp\left(-\frac{2x_{y-1}}{\beta_1}\right)\right)^{-1}\right] \quad (29)$$

and for the batches of sizes 1 *and* 2,

(a) *quantity premium if* $\beta_2 = 2\beta_1$, *and*
(b) *quantity discount if* $\beta_2 \leqslant \beta_1(2 - \ln 2)$.

The level of refinement in bound (29) can be seen by comparing its maximum value of $\beta_1 \ln 2$ to the maximum value $\bar{a} = \beta_1$ of $a(x)$ in (26). The difference is only by the factor of $\ln 2 = 0.69315$. The bound is tight since (29) holds as equality for $y = 2$ and $\lambda_2 = 0$. This result, under conditions of Theorem 3, shows that $\bar{p}_1(t, y) \geqslant \bar{p}_2(t, y)$ for all t, y if $\beta_2/\beta_1 \leqslant 2 - \ln 2 = 1.30685$ and $\bar{p}_1(t, y) \leqslant \bar{p}_2(t, y)$ if $\beta_2/\beta_1 = 2$.

In online appendix §EC.9, we present numerical experiments to refine the insights on perceived quantity premiums. The experiments indicate a complex interplay between concavity/convexity properties of the value function and stochastic order properties of the WTP distributions. This complexity arises because the stochastic order of unit WTP has two opposing influences on unit prices: directly via the hazard rate in the optimality conditions and indirectly through the convexity properties of the value function. Moreover, the persistent (for all t and y in the experiment) perceived quantity discounts are seen only in the region of inputs with low β_2/β_1 and β_3/β_2 ratios. The persistent premiums for sizes two and three occur in regions of, respectively, high β_2/β_1 but low λ_2/λ_1 and high β_3/β_2 but low λ_3/λ_2. In other regions of the parameter space, both quantity premiums and discounts are possible as t and y vary.

7. Concluding Remarks

Although our work focuses on the analysis and insights for a general problem of dynamic nonlinear pricing, the proposed model and its analysis can be extended to broader settings. In the current model, we consider a situation when a customer requires a fixed amount of product; i.e., he/she does not face a choice between batches of different size. What if there is a choice? A natural extension of the model would consider a situation when the market segmentation between batches of different size is imperfect, and each customer makes a choice between batches of different size. In the short term, customers may also behave strategically by trying to fill the desired order through several purchases of smaller orders made at different times. This results in a dynamic pricing problem with batch demand in the presence of consumer choice. Finally, in a dynamic pricing problem on a network, a study of break-up monotonicity may give an opportunity to better understand the structure of this important and complex problem.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/opre.2014.1285.

Acknowledgments

This research was supported in part by Natural Sciences and Engineering Research Council of Canada [Grants 261512-2009 and 341412-2011] and Queen's School of Business. The authors thank the department and associate editors, and the referees for their constructive suggestions, which helped to explain the model and improve the results. The second author sends prayers of utmost gratitude to God in whom he sought inspiration during this work.

Appendix A. Mathematical Proofs

A.1. Facts About the $\rho(x)$, $\delta(x)$, and $\xi(x)$

Consider a WTP random variable W described by the survival function $\pi(x)$, density function $f(p)$, and hazard rate $h(p)$ and assume that this distribution satisfies Assumptions 1–3. This section presents a discussion of the profit maximization problem with (inverse) demand function $\pi^{-1}(d)$ and cost x

$$\rho(x) = \max_{d\in[0,1]} d\pi^{-1}(d) - xd, \quad (30)$$

and properties of its optimal value $\rho(x)$, optimal demand $\delta(x)$ and the corresponding optimal price $\xi(x) = \pi^{-1}(\delta(x))$ as functions of x. The marginal revenue function (the derivative of $r(d) = d\pi^{-1}(d)$) has the form $J(d) = \pi^{-1}(d) - d/f(\pi^{-1}(d))$ and its reparametrization in terms of $p = \pi^{-1}(d)$—the form $\tilde{J}(p) = p - (h(p))^{-1}$.

Lemma 7. (a) *For any* $x \geqslant 0$, *there exists the unique optimal solution* $\delta(x) \in (0, 1)$ *to* (30) *given by equation* $J(\delta(x)) = x$. *The corresponding optimal price is given by the equation*

$$\tilde{J}(\xi(x)) = \xi(x) - (h(\xi(x)))^{-1} = x. \quad (31)$$

(b) *We have* $\delta'(x) = 1/J'(\delta(x))$ *and* $\delta''(x) = -J''(\delta(x))/(J'(\delta(x)))^3$ *for any* $x \geqslant 0$. *Therefore,* $\delta(x)$ *is a strictly decreasing function of* x. *It is convex in* x *if and only if* $J(d)$ *is convex.*

(c) *We have* $\rho'(x) = -\delta(x)$ *and* $\rho''(x) = -\delta'(x)$ *for any* $x \geqslant 0$. *Therefore,* $\rho(x)$ *is a strictly decreasing and strictly convex function of* x.

(d) $\rho(x) = \delta(x)(\xi(x) - x) = \pi(\xi(x))/h(\xi(x))$.

(e) $\lim_{x\to\infty} \xi(x) = \infty$, $\lim_{x\to\infty} \delta(x) = 0$ *and* $\lim_{x\to\infty} \rho(x) = 0$.

(f) $\xi(x)$ *is a strictly increasing function of* x *with* $\xi'(x) = (1 + h'(\xi(x))/(h(\xi(x)))^2)^{-1}$. *Therefore,* $\xi(x)$ *is convex if and only if* $1/h(p)$ *is convex.*

We can expand part (d) of the above lemma to obtain expressions for $\rho(x)$ only in terms of the hazard rate and the optimal price function $\xi(x)$.

Lemma 8. *The following representations hold*

$$\rho(x) = \frac{\exp(-\int_0^{\xi(x)} h(\theta)\, d\theta)}{h(\xi(x))}, \tag{32}$$

$$\rho(x) = \rho(x_0) \exp\left(-\int_{x_0}^{x} h(\xi(\theta))\, d\theta\right). \tag{33}$$

Properties of $\rho(x)$ help us prove that $x_1 \to \infty$ as $t \to \infty$. Consider a differential equation for x_1 (a special case of (23) for $y = 1$) where we omit subscripts for brevity:

$$\dot{x} = \lambda \rho(x) \quad \text{with initial condition } x(0) = 0. \tag{34}$$

Lemma 9. *If* $\lambda > 0$ *and* $x(t)$ *is the solution to* (34), *then* $\lim_{t\to\infty} x(t) = \infty$.

Additional assumption on the growth of hazard rates implies the following result:

Lemma 10. *Assumption* 4 *yields*

$$\frac{\bar{\gamma}}{x} \leqslant \frac{h'(x)}{h(x)} \leqslant \frac{\gamma}{x}. \tag{35}$$

A.2. Proof of Proposition 1

Strict concavity of the objective is immediate from Assumption 3, and the feasible set is obviously convex. The problem also satisfies Slater's condition. Thus, the first-order optimality (KKT) conditions are necessary and sufficient. Uniqueness follows from strict concavity of the objective function. It remains to show that the solution to the subproblem in (3) exists and satisfies KKT conditions. Indeed, $V(t, y) - V(t, y-i) \geqslant 0$, and the range of $J_i(d)$ includes all nonnegative values (since $\lim_{d\to 0} J_i(d) = \lim_{p\to\infty} \tilde{J}_i(p) = \infty$). Therefore, solution to (18) exists and belongs to the interval $(0, 1)$. Finally, (18) represents KKT conditions for the case where all Lagrange multipliers are zero.

A.3. Proof of Corollary 3

Inequality $p_i(t, y) \leqslant p_{i'}(t, y)$ holds because $\tilde{J}_i(p_i(t, y)) = V(t, y) - V(t, y-i) \leqslant V(t, y) - V(t, y-i') = \tilde{J}_{i'}(p_{i'}(t, y)) \leqslant \tilde{J}_i(p_{i'}(t, y))$ (where the last inequality follows from Lemma 1) and $\tilde{J}_i(p_i)$ is increasing (as mentioned earlier, this follows from Assumption 3).

A.4. Proof of Theorem 1

From (21), it follows that $p_i(t, y) \leqslant (\geqslant) \sum_{j=0}^{k} p_{i_j}(t, y_j)$ if and only if

$$(h_i(p_i(t, y)))^{-1} \leqslant (\geqslant) \sum_{j=0}^{k} (h_{i_j}(p_{i_j}(t, y_j)))^{-1}.$$

In part (a), suppose that $p_i(t, y) > \sum_{j=0}^{i-1} p_1(t, y-j)$. Then

$$h_i(p_i(t,y)) \geqslant h_i\left(\sum_{j=0}^{i-1} p_1(t,y-j)\right) \geqslant \frac{1}{i} h_1\left(\frac{1}{i}\sum_{j=0}^{i-1} p_1(t,y-j)\right),$$

where the first inequality follows from Assumption 3 and the second from $(1/i)W_i \leqslant_{\text{hr}} W_1$. Thus,

$$\begin{aligned}(h_i(p_i(t, y)))^{-1} &\leqslant i\left(h_1\left(\frac{1}{i}\sum_{j=0}^{i-1} p_1(t, y-j)\right)\right)^{-1} \\ &\leqslant \sum_{j=0}^{i-1} (h_1(p_1(t, y-j)))^{-1}\end{aligned}$$

(where the last inequality follows from the convexity of $(h_1(p_1))^{-1}$), a contradiction.

In part (b), suppose that $p_i(t, y) > \sum_{j=0}^{k} p_{i_j}(t, y_j)$ and observe that $\sum_{j=0}^{k} (i_j/i) = 1$. By applying convexity property of $(h_i(p_i))^{-1}$ before stochastic order conditions $(1/i)W_i \leqslant_{\text{hr}} (1/i_j)W_{i_j}$, we obtain

$$\begin{aligned}(h_i(p_i(t, y)))^{-1} &\leqslant \left(h_i\left(\sum_{j=0}^{k} p_{i_j}(t, y_j)\right)\right)^{-1} \\ &= \left(h_i\left(\sum_{j=0}^{k} \frac{i_j}{i} \frac{i p_{i_j}(t, y_j)}{i_j}\right)\right)^{-1} \\ &\leqslant \sum_{j=0}^{k} \frac{i_j}{i}\left(h_i\left(\frac{i p_{i_j}(t, y_j)}{i_j}\right)\right)^{-1} \\ &\leqslant \sum_{j=0}^{i-1} \frac{i_j}{i}\left(\frac{i_j}{i} h_{i_j}(p_{i_j}(t, y_j))\right)^{-1} \\ &= \sum_{j=0}^{i-1} (h_{i_j}(p_{i_j}(t, y_j)))^{-1};\end{aligned}$$

again, a contradiction.

Proofs of parts (c) and (d) are similar to those of (a) and (b), respectively.

A.5. Proof of Proposition 2

We begin by showing the first statement that involves conditions (24) and (25) for specific value of y. Since $V(0, y) = 0$, for all $y \geqslant 0$, observe from (20) that $(\partial/\partial t)V(t, y)|_{t=0} = \sum_{i=1}^{y} \lambda_i \rho_i(0)$. Therefore,

$$\begin{aligned}\frac{\partial}{\partial t}[V(t, y) - V(t, y-1)]_{t=0} &= \lambda_y \rho_y(0) > (<) \lambda_{y+1}\rho_{y+1}(0) \\ &= \frac{\partial}{\partial t}[V(t, y+1) - V(t, y)]_{t=0}\end{aligned}$$

whenever (24) or (25) holds. Function $V(t, \cdot)$ is continuously differentiable in t as the solution to the boundary-value problem for (20). Therefore, this strict inequality between derivatives will hold in some small neighborhood of $t = 0$. The first claim of the proposition follows.

Suppose $\bar{y} = Y$. We can linearize each Equation (23) in the neighborhood of zero leading to

$$\begin{aligned}\dot{x}_y &= \sum_{i=1}^{y} \lambda_i \big[\rho_i(0) + \rho_i'(0)(x_y + \cdots + x_{y-i+1})\big] \\ &\quad - \sum_{i=1}^{y-1} \lambda_i \big[\rho_i(0) + \rho_i'(0)(x_{y-1} + \cdots + x_{y-i})\big] + o_y(x),\end{aligned}$$

where $o_y(x)$ represents the terms that vanish superlinearly as $x \to 0$ and depends only on those members of collection $x_1, \ldots, x_y$ that explicitly appear in the right-hand side of (23). Collecting the terms with the same i, we obtain

$$\dot{x}_y = \lambda_y[\rho_y(0) + \rho_y'(0)(x_y + \cdots + x_1)] + \sum_{i=1}^{y-1} \lambda_i \rho_i'(0)(x_y - x_{y-i}) + o_y(x).$$

The constant term $\lambda_y \rho_y(0)$ dominates all other terms since it is a positive constant, the terms linear in $x_1, \ldots, x_y$ and are $o(1)$, and the term $o_y(x)$ is even smaller near $t = 0$. Thus, we obtain

$$x_y(t) = \lambda_y \rho_y(0) t + o(t). \tag{36}$$

The coefficient $\lambda_y \rho_y(0)$ in front of t is precisely the expression that appears in inequalities (24) and (25). Because these inequalities are strict and $o(t)$ terms vanish superlinearly in t as $t \to 0$, the concavity (convexity) claims of the proposition follow if $\bar{y} = Y$.

It remains to provide a refinement of the concavity claim when $\bar{y} < Y$. The proof is by induction. To establish the base case for the induction, we use the fact that (36) holds for $y = 1, \ldots, \bar{y}$. Thus, x_y is decreasing for $y = 1, \ldots, \bar{y}$. Suppose now that

$$x_{n\bar{y}+i}(t) = b_{n\bar{y}+i} t^{n+1} + o(t^{n+1}), \quad i = 1, \ldots, \bar{y} \tag{37}$$

holds for some $n \geqslant 0$ and values $b_{n\bar{y}+i}$ are positive and strictly decreasing in $i = 1, \ldots, \bar{y}$. By induction, we show that representation (37) also holds for $n + 1$. Within this inductive step on n we also employ induction on i. Indeed, let $1 \leqslant j \leqslant \bar{y}$, $y = (n+1)\bar{y} + j$ and suppose that (37) holds for $n + 1$ and $i = 1, \ldots, j - 1$. Consider $\dot{x}_y$ that, because $\lambda_i = 0$ for $i > \bar{y}$, satisfies

$$\dot{x}_y = \sum_{i=1}^{\bar{y}} \lambda_i \rho_i'(0)(x_y - x_{y-i}) + o_y(x). \tag{38}$$

The higher-order term $o_y(x)$ depends only on $x_{y-i}(t)$, $i = 0, \ldots, \bar{y}$ which are all $O(t^{n+1})$. Thus, $o_y(x)$ is $o(t^{n+1})$. It follows that $x_y(t)$ is $o(t^{n+1})$. Note that, by induction on i within the inductive step on n, we have $x_{y'}(t) = o(t^{n+1})$ for $y' = (n+1)\bar{y} + 1, \ldots, y - 1$. Collecting all of the $o(t^{n+1})$ terms in (38) as a single $o(t^{n+1})$ term and using the inductive assumption for the remaining terms (i.e., $x_{y-i}(t) = b_{y-i} t^n + o(t^{n+1})$, $i = y - (n+1)\bar{y}, \ldots, \bar{y}$), we obtain

$$\dot{x}_y = -\sum_{i=y-(n+1)\bar{y}}^{\bar{y}} \lambda_i \rho_i'(0) b_{y-i} t^{n+1} + o(t^{n+1}).$$

After integrating this equation over t, we obtain $x_y(t) = b_y t^{n+2} + o(t^{n+2})$ where

$$b_y = -\frac{1}{n+2} \sum_{i=y-(n+1)\bar{y}}^{\bar{y}} \lambda_i \rho_i'(0) b_{y-i}$$

is positive because $\rho_i'(0)$ is negative for all i. Also, if $y > (n+1)\bar{y} + 1$, we have

$$b_y < b_{y-1} = -\frac{1}{n+2} \sum_{i=y-(n+1)\bar{y}-1}^{\bar{y}} \lambda_i \rho_i'(0) b_{y-i-1}$$

because $b_{y-i} < b_{y-i-1}$ and the summation for b_{y-1} includes one more term than for b_y.

A.6. Proof of Theorem 2

To simplify the notation, consider, without loss of generality, batches of any size up to Y but set the arrival rates $\lambda_i \equiv 0$ for $i > \bar{y}$.

The derivative of (20) in t is

$$\frac{\partial^2}{\partial t^2} V(t, y) = -\sum_{i=1}^{y} \lambda_i \delta_i (V(t, y) - V(t, y-i)) \cdot \left[\frac{\partial V(t, y)}{\partial t} - \frac{\partial V(t, y-i)}{\partial t}\right]. \tag{39}$$

Note that $\partial^2 V(t, y)/\partial t^2 \leqslant 0$ if all the brackets $[\partial V(t, y)/\partial t - \partial V(t, y-i)/\partial t] \geqslant 0$ since $\delta_i(\cdot) \geqslant 0$ for any $i = 1, \ldots, y$. Consider $[\cdot]$ in (39) for the case with $i = 1$.

$$\begin{aligned}
\dot{x}_y &= \frac{\partial V(t, y)}{\partial t} - \frac{\partial V(t, y-1)}{\partial t} \\
&= \lambda_y \rho_y(V(t, y)) + \sum_{i=1}^{y-1} \lambda_i \rho_i (V(t, y) - V(t, y-i)) \\
&\quad - \sum_{i=1}^{y-1} \lambda_i \rho_i (V(t, y-1) - V(t, y-i-1)) \\
&= \lambda_y \rho_y(V(t, y)) + \sum_{i=1}^{y-1} \lambda_i \big[\rho_i(V(t, y) - V(t, y-i)) \\
&\qquad - \rho_i(V(t, y-1) - V(t, y-i-1))\big].
\end{aligned}$$

Denote $z_y \triangleq V(t, y) - V(t, y-i) = \sum_{j=1}^{i-1} x_{y-j} + x_y$ and $z_{y-1} \triangleq V(t, y-1) - V(t, y-i-1) = \sum_{j=1}^{i-1} x_{y-j} + x_{y-i}$. Because of the concavity of $V(t, i)$ in i, we have $x_y \leqslant x_{y-i}$ for any $i = 1, \ldots, y-1$. Therefore, $z_y \leqslant z_{y-1}$ and (by Lemma 7 (c)), $\rho_i(z_y) \geqslant \rho_i(z_{y-1})$, implying $\dot{x}_y \geqslant 0$. The proof is the same for $i = 2, \ldots, y$ in (39). Parts (c) and (d) are immediate.

A.7. Main Lemmas Used in the Proof of Theorem 3

Throughout this section, we assume that Assumptions 1–6 hold. Function $a(x)$ defined in (26) is used extensively in our analysis and possesses several key properties:

Lemma 11. *The function $a(x)$ has the following properties*:
(a) $a(x) \geqslant 0$ *for all* $x \geqslant 0$,
(b) $a(0) = 0$,
(c) $a(x) \leqslant ((1+\gamma)/h_1(\xi_1(x)))[1 - \exp(-\int_{\xi_1(0)}^{\xi_1(x)} h_1(\theta)\, d\theta)]$,
(d) $\sup_{x \geqslant 0} a(x) < \bar{a} = \xi_1(0) \exp(1 - [\xi_1(0) \sup_z h_1(z)]^{-1})$, *and*
(e) $\lim_{x\to\infty} a(x) = 0$ *if* $\lim_{x\to\infty} h_1(x) = \infty$.

This lemma provides a very specific upper bound (c) for the value of $a(x)$ at given x. This bound is essentially in terms of the hazard rate $h_1(x)$ since $\xi_1(x)$ is defined via $h_1(x)$ as a solution to (31). Moreover, there is a global upper bound given by (d) when $h_1(x)$ is bounded and an observation (e) that $a(x)$ tends to zero as $x \to \infty$ when $h_1(x)$ is unbounded.

The following lemma stipulates sufficient conditions for the trajectory of the system that guarantee upper bound on the difference between consecutive opportunity costs.

Lemma 12. *Suppose that opportunity costs satisfy inequalities* $x_1 \geqslant \ldots \geqslant x_{y+1}$ *and*

$$\lambda_{i+1} \rho_{i+1}(x_y + \cdots + x_{y-i}) \leqslant \lambda_i \rho_i (x_{y-1} + \cdots + x_{y-i}), \quad i = 1, \ldots, y \wedge \bar{y} - 1, \tag{40}$$

for all $t \geqslant 0$; *then* $x_y - x_{y+1} \leqslant a(x_y)$.

The set of inequalities in (40) becomes empty when $y=1$. This special case of Lemma 12 is used in the base case of our inductive proof (in fact, only Equation (42) is needed in the base case).

The next lemma plays a key role in translating Assumptions 4–6 on the inputs into inequalities that hold on the trajectory of the system and can be used in the inductive proof. Parts (a) and (b) are used at different points of the main inductive step.

Lemma 13. *Suppose $v_0 \geqslant v_1 \geqslant v_2 \geqslant \cdots \geqslant v_i \geqslant 0$ and $i<\bar{y}$.*

(a) *If $v_j - v_{j+1} \leqslant a(v_j)$, $j=1\ldots i$, then*

$$\lambda_{i+1}\rho_{i+1}\left(v_{i+1}+\sum_{j=1}^{i} v_j\right) < \lambda_i\rho_i\left(\sum_{j=1}^{i} v_j\right). \tag{41}$$

Moreover, the natural log of the ratio of the left- to the right-hand side in inequality (41) *is bounded from above by* $\ln[\lambda_{i+1}\rho_{i+1}(0)/\lambda_i\rho_i(0)]+\Phi_i$.

(b) *If $v_j - v_{j+1} \leqslant a(v_j)$, $j=1\ldots i-1$, then*

$$\lambda_{i+1}\rho_{i+1}\left(2v_i+\sum_{j=1}^{i-1} v_j\right) < \lambda_i\rho_i\left(v_i+\sum_{j=1}^{i-1} v_j\right). \tag{42}$$

$$\lambda_{i+1}\left[\rho_{i+1}\left(2v_i+\sum_{j=1}^{i-1} v_j\right)-\rho_{i+1}\left(v_i+\sum_{j=1}^{i-1} v_j+v_0\right)\right]$$
$$\leqslant \lambda_i\left[\rho_i\left(v_i+\sum_{j=1}^{i-1} v_j\right)-\rho_i\left(\sum_{j=1}^{i-1} v_j+v_0\right)\right] \tag{43}$$

where inequality (43) *is strict unless $v_0=\cdots=v_i$. Moreover, the natural log of the ratio of the left- to the right-hand side in inequality* (43) *is bounded from above by* $\ln[\lambda_{i+1}\rho_{i+1}(0)/\lambda_i\rho_i(0)]+\Phi_i+\ln\max\{1,M_i\}$.

A.8. Proof of Theorem 3

The proof is by induction on the inventory level, and its plan is as follows. In the base case, the maximum inventory level is two. We establish $x_1\geqslant x_2$ using assumptions on the inputs (only need $i=1$ in Assumptions 5, 6 as well as in Lemma 13(b)). We then proceed by showing that $x_1-x_2\leqslant a(x_1)$ (directly using Lemma 12 for $y=1$, without resorting to Lemma 13(a)). In the general case, we assume that $x_1\geqslant x_2\geqslant x_3\geqslant\ldots\geqslant x_y$ and $x_i-x_{i+1}\leqslant a(x_i)$, $i=1,\ldots,y-1$. We use i up to y in Assumptions 5 and 6 as well as in Lemma 13(b) to establish $x_y\geqslant x_{y+1}$. We proceed by using this fact along with the inductive assumptions and Lemma 13(b) to establish $x_y-x_{y+1}\leqslant a(x_y)$.

We start the proof of base case by showing that $x_1\geqslant x_2$. The value function is continuously differentiable, and we consider set $\mathcal{T}_1 \triangleq \{t\geqslant 0\colon\ x_2(t')<x_1(t')\ \forall\, t'\in(0,t)\}$ and its upper bound $\hat{t}_1\triangleq\sup\mathcal{T}_1$. If $\hat{t}_1=\infty$, then the base case is established. Suppose that $\hat{t}_1<\infty$. Because of (42), Proposition 2 implies that $\hat{t}_1>0$. It must be true that $\{\dot{x}_2-\dot{x}_1\}|_{t=\hat{t}_1}\geqslant 0$, but we demonstrate the opposite, arriving at contradiction. Since x_1 and x_2 are continuous, we have $x_2(\hat{t}_1)=x_1(\hat{t}_1)$. Equation (23) for $y=1,2$ at $t=\hat{t}_1$ leads to

$$\dot{x}_2(\hat{t}_1)=\lambda_2\rho_2(2x_1)\quad s(\text{terms with } \rho_1(x_1) \text{ cancel}),$$

$$\dot{x}_1(\hat{t}_1)=\lambda_1\rho_1(x_1).$$

Using Lemma 13(b) with $i=1$ and $v_1=x_1$, inequality (42) implies

$$\{\dot{x}_2-\dot{x}_1\}|_{t=\hat{t}_1}=\lambda_2\rho_2(2x_1)-\lambda_1\rho_1(x_1)<0.$$

The latter inequality also holds if $\lambda_2=0$.

A lower bound for x_2 in terms of x_1 results from Lemma 12 for $y=1$ (in this case, the set of inequalities in (40) is empty).

Suppose that the statement holds up to some level of inventory $y\geqslant 1$ and establish it for $y+1$. Consider a set $\mathcal{T}_y\triangleq\{t\geqslant 0\colon x_{y+1}(t')<x_y(t')\ \forall\, t'\in(0,t)\}$ and $\hat{t}_y\triangleq\sup\mathcal{T}_y$. By the inductive assumption, we have $x_1\geqslant\cdots\geqslant x_y$ for all $t\geqslant 0$ (with strict inequality for $t>0$). By the definition of $\hat{t}_y$ and continuity of the solution we have $x_{y+1}(\hat{t}_y)=x_y(\hat{t}_y)$. Consider $y<\bar{y}$ first. Using (23), we get

$$\dot{x}_{y+1}(\hat{t}_y)=\sum_{i=1}^{y+1}\lambda_i\rho_i\left(x_y+\sum_{j=0}^{i-2}x_{y-j}\right)-\sum_{i=1}^{y}\lambda_i\rho_i\left(\sum_{j=0}^{i-1}x_{y-j}\right),$$

$$\dot{x}_y(\hat{t}_y)=\sum_{i=1}^{y}\lambda_i\rho_i\left(\sum_{j=0}^{i-1}x_{y-j}\right)-\sum_{i=1}^{y-1}\lambda_i\rho_i\left(\sum_{j=1}^{i}x_{y-j}\right).$$

Consider $\{\dot{x}_{y+1}-\dot{x}_y\}_{t=\hat{t}_y}$. The terms corresponding to $i=1$ in $\dot{x}_{y+1}(\hat{t}_y)$ cancel, and we can collect the remaining terms so that

$$\begin{aligned}\{\dot{x}_{y+1}-\dot{x}_y\}_{t=\hat{t}_y}&=\lambda_{y+1}\rho_{y+1}\left(2x_y+\sum_{j=1}^{y-1}x_{y-j}\right)-\lambda_y\rho_y\left(x_y+\sum_{j=1}^{y-1}x_{y-j}\right)\\&\quad+\sum_{i=1}^{y-1}\left\{\lambda_{i+1}\left[\rho_{i+1}\left(2x_y+\sum_{j=1}^{i-1}x_{y-j}\right)-\rho_{i+1}\left(x_y+\sum_{j=1}^{i-1}x_{y-j}+x_{y-i}\right)\right]\right.\\&\qquad\left.-\lambda_i\left[\rho_i\left(x_y+\sum_{j=1}^{i-1}x_{y-j}\right)-\rho_i\left(\sum_{j=1}^{i}x_{y-j}\right)\right]\right\}.\end{aligned}$$

Inequality (43) in Lemma 13(b), applied for $i=1,\ldots y-1$ with $v_i=x_y$, $v_{i-1}=x_{y-1},\ldots,v_1=x_{y-i+1}$, $v_0=x_{y-i}$, yields the inequalities

$$\lambda_{i+1}\left[\rho_{i+1}\left(2x_y+\sum_{j=1}^{i-1}x_{y-j}\right)-\rho_{i+1}\left(x_y+\sum_{j=1}^{i-1}x_{y-j}+x_{y-i}\right)\right]$$
$$\leqslant\lambda_i\left[\rho_i\left(x_y+\sum_{j=1}^{i-1}x_{y-j}\right)-\rho_i\left(\sum_{j=1}^{i-1}x_{y-j}+x_{y-i}\right)\right],$$

implying that $\{\dot{x}_{y+1}-\dot{x}_y\}_{t=\hat{t}_y}\leqslant\lambda_{y+1}\rho_{y+1}(2x_y+\sum_{j=1}^{y-1}x_{y-j})-\lambda_y\rho_y(x_y+\sum_{j=1}^{y-1}x_{y-j})$, which is negative by inequality (42) in Lemma 13(b), a contradiction. If $y\geqslant\bar{y}$, the above expression has fewer terms because the arrival rates are zero for packages of any size larger than $\bar{y}$. However, each of the remaining terms is strictly negative. Thus, we still arrive at a contradiction.

The final element of the inductive step is the application of Lemma 12 where we use Lemma 13(a) with $i=1,\ldots,\min\{y-1,\ \bar{y}-1\}$ and $v_i=x_y$, $v_{i-1}=x_{y-1},\ldots,v_1=x_{y-i+1}$, $v_0=x_{y-i}$ to verify conditions (40).

The uniform bound $\bar{a}$ on $a(x_y)$ in part (b) follows from Lemma 11(d). In addition, Lemma 9 establishes that $x_1\to\infty$ as $t\to\infty$. These results along with inequality (26) imply that all opportunity costs tend to infinity as long as the time horizon increases and that the difference between consecutive opportunity costs tends to zero if $h_1(x)$ is unbounded.

Parts (c) and (d) immediately follow from part (a).

A refinement to monotonicity results of this theorem can be obtained for the special case when the product is only sold in units or in pairs (that is, $\bar{y}=2$):

COROLLARY 5. *For* $\bar{y}=2$, *sufficient conditions for Theorem* 3(a) *can be relaxed by using* $\Phi_1=0$ *in Assumption* 6. *Part* (b) *can be established separately from part* (a) *but still requires the original value of* Φ_1 *in Assumption* 6.

This result points to the role of part (b) in making the requirements of Assumption 6 more stringent. The proof of this corollary is in the online appendix.

A.9. Proof of Proposition 3

Suppose that unit opportunity cost for both batch sizes is equal to w. Equation $\tilde{J}_k(\xi_k(x))=x$ in the form $\xi_k(kw)-(h_k(\xi_k(kw)))^{-1}=kw$ gives

$$\frac{\xi_k(kw)}{k}-(kh_k(\xi_k(kw)))^{-1}=w, \quad \text{and} \tag{44}$$

$$\frac{\xi_{k'}(k'w)}{k'}-(k'h_{k'}(\xi_{k'}(k'w)))^{-1}=w. \tag{45}$$

By definition, the order $(1+\Delta)(W_k/k)\leqslant_{\text{hr}} W_{k'}/k'$ means that for any z

$$\frac{kh_k(kz/(1+\Delta))}{1+\Delta}\geqslant k'h_{k'}(k'z). \tag{46}$$

Suppose that

$$\frac{\xi_k(kw)}{k}\geqslant\frac{\xi_{k'}(k'w)}{k'}. \tag{47}$$

Then relations (44)–(45) result in $(kh_k(\xi_k(kw)))^{-1}\geqslant(k'h_{k'}(\xi_{k'}(k'w)))^{-1}$ or $kh_k(\xi_k(kw))\leqslant k'h_{k'}(\xi_{k'}(k'w))$. Using (46) with $z=\xi_{k'}(k'w)/k'$, inequality (47) and that h_k is increasing in the right-hand side of the last inequality results in

$$k'h_{k'}(\xi_{k'}(k'w))\leqslant\frac{kh_k(k\xi_{k'}(k'w)/((1+\Delta)k'))}{1+\Delta}\leqslant\frac{kh_k(\xi_k(kw)/(1+\Delta))}{1+\Delta}<kh_k(\xi_k(kw)),$$

a contradiction.

References

Araman V, Caldentey R (2009) Dynamic pricing for nonperishable products with demand learning. *Oper. Res.* 57(5):1169–1188.

Baricz Á (2008) Mill's ratio: Monotonicity patterns and functional inequalities. *J. Math. Anal. Appl.* 340(2):1362–1370.

Besbes O, Zeevi A (2009) Dynamic pricing without knowing the demand function: Risk bounds and near-optimal algorithms. *Oper. Res.* 57(6):1407–1420.

Bolton L, Warlop L, Alba J (2003) Consumer perceptions of price (un)fairness. *J. Consumer Res.* 29(4):474–491.

Bremaud P (1981) *Point Processes and Queues* (Springer-Verlag, New York).

Dhebar A, Oren S (1986) Dynamic nonlinear pricing in networks with interdependent demand. *Oper. Res.* 34(3):384–394.

Dong L, Kouvelis P, Tian Z (2009) Dynamic pricing and inventory control of substitute products. *Manufacturing Service Oper. Management* 11(2):317–339.

Elmaghraby W, Gülcü A, Keskinocak P (2008) Designing optimal preannounced markdowns in the presence of rational customers with multi-unit demands. *Manufacturing Service Oper. Management* 10(1): 126–148.

Farias V, Van Roy B (2010) Dynamic pricing with a prior on market response. *Oper. Res.* 58(1):16–29.

Gallego G, van Ryzin G (1994) Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Sci.* 40(8): 999–1020.

Gallego G, van Ryzin G (1997) A multiproduct dynamic pricing problem and its application to network yield management. *Oper. Res.* 45(1): 24–41.

Haws K, Bearden W (2006) Dynamic pricing and consumer fairness perceptions. *J. Consumer Res.* 33(3):304–311.

Kleywegt A, Papastavrou J (2001) The dynamic and stochastic knapsack problem with random sized items. *Oper. Res.* 49(1):26–41.

Laffont J, Martimort D (2001) *The Theory of Incentives: The Principal-Agent Model* (Princeton University Press, Princeton, NJ).

Lee T, Hersh M (1993) A model for dynamic airline seat inventory control with multiple seat bookings. *Transportation Sci.* 27(3): 252–256.

Maglaras C, Meissner J (2006) Dynamic pricing strategies for multiproduct revenue management problems. *Manufacturing Service Oper. Management* 8(2):136–148.

Mierendorff K (2011) Optimal dynamic mechanism design with deadlines. Working paper, University of Zurich, Switzerland.

Myerson B (1981) Optimal auction design. *Math. of Oper. Res.* 6(1): 58–73.

Shaked M, Shanthikumar J (1994) *Stochastic Orders and Their Applications* (Academic Press Inc., Boston).

Talluri K, van Ryzin G (2004) *The Theory and Practice of Revenue Management* (Kluwer Academic Publishers, Norwell, MA).

van Slyke R, Young Y (2000) Finite horizon stochastic knapsacks with applications to yield management. *Oper. Res.* 48(1):155–172.

Wilson R (1993) *Nonlinear Pricing* (Oxford University Press, New York).

Xu X, Hopp W (2009) Price trends in a dynamic pricing model with heterogeneous customers: A martingale perspective. *Oper. Res.* 57(5): 1298–1302.

Zhang D, Cooper W (2009) Pricing substitutable flights in airline revenue management. *Eur. J. Oper. Res.* 197(3):848–861.

Yuri Levin is a distinguished professor of operations management and director of master of management analytics program at Queen's School of Business in Kingston, Canada. He has developed innovative approaches and published widely in the general area of revenue management and dynamic pricing.

Mikhail Nediak is an assistant professor at Queen's School of Business in Kingston, Canada. His current research interests belong to the rich area of revenue management and dynamic pricing.

Andrei Bazhanov is a research fellow at Queen's School of Business in Kingston, Canada. His main research interests are dynamic pricing and revenue management with strategic consumers.