SELLING PASSES TO STRATEGIC CUSTOMERS

JUE WANG, YURI LEVIN AND MIKHAIL NEDIAK

Abstract. Passes are prepaid packages of multiple units of goods or services, typically with flexible delivery or consumption dates. They can take a variety of forms, such as commuter passes in transportation or capped quota in telecommunication, as well as memberships in resorts, health, or beauty clubs. We consider a monopolist selling a fixed capacity by dynamically pricing passes as well as individual units in the presence of strategic customers. The pass contains a finite number of redeemable credits that will expire at the end of the sales horizon, and the customers are strategic in both purchasing and using the pass. Using optimal control theory, we find that the optimal pricing policy has a turnpike property; the optimal price trajectories stay near the steady state for most of the sales horizon, and fixed pricing policies perform very well. We show that passes should offer quantity discounts as long as customers are not fully strategic. In the turnpike, the pass generates a higher revenue rate than individual sale if and only if the capacity is limited and the customers are not fully strategic. Furthermore, the quantity discounts offered by passes induce advance purchase, creating a new mechanism to mitigate and even capitalize on strategic behavior in certain circumstances.

1. Introduction

In diverse industries such as airline, rail, public transportation, resorts, and theme parks, the majority of customers are now familiar with passes. Examples in airlines include the AAirpass offered by American Airlines, PassPlus by United Airlines, commuter pass by Air Canada, airpass by Star Alliance. Other examples are Greyhound's 20Pak pass, Disney World seasonal pass, and Flex pack for National Basketball Association (NBA), Major League Baseball (MLB). These passes are prepaid at a flat rate and grant a holder a certain (possibly unlimited) access to services before an expiration date. Passes can also take other forms, such as packs of tokens, vouchers, store value cards (smart cards), meal plans, and health/beauty club memberships.

In the airlines, a "flight pass" is a prepaid package of electronic flight credits that can be used for travel during a specific period. The pass permits its holder to obtain a seat on a specified number of flights at a fixed price when the pass is purchased. Passes allow the customers to lock in the price and offer considerable booking flexibility such as allowing customers to select any flight that has available capacity. While these features are quite attractive, purchasing a pass does not eliminate the risk of a lack of capacity on the preferred flights, or a failure to use all of the purchased flight credits by the expiration date.

Flight passes are typically sold and managed through the same booking systems as regular tickets. Although individual tickets are priced dynamically, most passes are sold at fixed prices. Airlines can potentially price passes dynamically but the models used in such pricing must (1) consider pass pricing in conjunction with dynamic pricing for individual flights, (2) incorporate customer choice behavior between passes and individual bookings, and (3) account for credit utilization and repurchases. This paper addresses the following research questions. Should the passes be priced dynamically? How far is the fixed price from optimality? How much improvement can dynamic pass pricing offer? How do strategic behavior and capacity affect the benefits of passes and resulting pricing policies?

Date: January 31, 2015.

Queen's School of Business, Queen's University, Kingston, Ontario, Canada, K7L 3N6.

Benefits of passes are affected by two types of cannibalization effects, inter-product and inter-temporal. The inter-product cannibalization takes place when passes reduce the revenue from regular tickets, e.g., some customers may be willing to pay a higher individual ticket price but instead end up using a pass credit to pay less for this product. This is one of the reasons many firms hesitate to introduce passes. However, passes can also attract customers who would otherwise not purchase a ticket. It becomes important to understand the impact of passes on the total revenue. Passes also result in inter-temporal cannibalization: a fixed number of credits creates a temporary drop in demand for both individual tickets and passes (similar to post-promotion dip, see Macé and Neslin (2004)) because pass holders have almost no incentives to make new purchases before spending all the pass credits. For unlimited passes, this drop is persistent. Large sales of passes may significantly decrease future demand. It is important to jointly price passes and single items under both types of cannibalization.

Customers are inherently forward-looking but uncertain about their future needs. When both passes and single items are priced dynamically, some unique challenges arise. For example, on December 4, 2014, the price for Air Canada 4-credit commuter pass (valid for one year) between New York and Toronto is \$664, which is equivalent to \$166 per trip. This price is higher than the spot price on December 17, 2014 (\$160) but is lower than on December 18, 2014, which is \$181. Strategic customers will try to compare all foreseeable future prices and make the purchase that maximizes their long-term utilities. Hence, the perceived value of passes increases if the future prices are higher because, in that case, passes offer more savings.

The credit utilization process is an integral part of customer behavior and should be considered while making price decisions. Intuitively, if the pass holder is going to spend all the pass credits, the issuer would prefer the credits to be spent as soon as possible, so that a customer is more likely to purchase again. On the other hand, if pass holders are not going to spend all credits, then the issuer can keep the unused credits as profit. The pass utilization process is affected by several factors, some of which are controllable by the issuer. For example, high future prices can slow down the credit utilization process, because pass holders tend to curb their consumption to avoid renewing passes (or purchasing new tickets) at high prices (Ailawadi and Neslin, 1998, Bell et al., 2002). In other words, strategic customers may time their consumption in a way that the pass can be renewed at lower prices. It is also known that packaging multiple products tends to accelerate consumption, see Wansink (1996), but the utilization rate drops as credits deplete. The latter phenomenon takes place because, as implied by the scarcity theory (see Folkes et al. (1993)), smaller quantities are often perceived as more valuable. Finally, pass holders may also consume at a higher rate close to the expiration date for fear of wasting unused credits. Ultimately, demand depends on how strategic the customers are.

A pass can be comprised of fixed or unlimited credits. In this paper, we focus on fixed credits. This is because an unlimited pass is a special case of a two-part tariff: the pass price is the lump-sum fee whereas the per-unit charge is zero (Carbajo, 1988). The unlimited pass holder has no incentive to repurchase before the expiration date. Therefore, a customer who purchases such a pass leaves the pool of potential buyers. The resulting consumption behavior is also different from the case with a finite number of credits, in which pass holders eventually return and repurchase. Unlimited passes are rare in industries using dynamic pricing. For example, American Airlines introduced the AAirpass program in the early 1980s granting unlimited life-time first-class travel for a flat fee of \$250,000. However, the program turned out to be unsuccessful because of abuse by heavy users (Bensinger, 2012). The airline later closed unlimited passes and switched to prepaid blocks of airmiles that can be redeemed for flights. In general, services with limited capacity tend to use finite credits whereas unlimited passes are mainly used for the cases of ample capacity (public transit, theme parks) or low marginal cost per unit. Thus, the models of the pass utilization process are substantially different for limited and unlimited passes.

Summary of results. In this study, we observe that the optimal pricing policy has the "turnpike" property: the optimal price trajectories stay close to a steady state (turnpike) most of the time except at the beginning and the end of the horizon, provided that the sales horizon is sufficiently long (to the extend that pass holders can spend all credits before the expiration date). Further, the revenue loss from a fixed-price policy does not increase with the sales horizon, indicating that its relative sub-optimality decreases as the sales horizon increases. This suggests that dynamic pricing of passes cannot significantly improve the revenue. After introducing the pass, even the regular price for individual sales need not be dynamic.

We also study the impact of strategic behavior and capacity on the turnpike revenue rate. In particular, passes offer quantity discounts and encourage consumption as long as customers are not fully strategic, which is consistent with practice and intuition. What is less intuitive is that the pass credit generates a higher turnpike revenue rate than a regular sale when customers are not fully strategic *and* the capacity is limited, in which case the capacity scarcity also makes an appreciable contribution to the revenue.

Perhaps the most significant advantage of the pass is that it allows the issuer to benefit from strategic behavior. In prior results, strategic customer behavior typically decreases the revenue. However, this trend can be reversed in the presence of passes: more strategic customers generate higher revenues. Passes do so by offering a quantity discount for advance purchase. Such combination discourages strategic waiting behavior because the customers must make advance purchase to enjoy this discount. Unlike capacity rationing (see Liu and van Ryzin (2008) and Su (2007)), passes do not create shortage risk, representing a new mechanism for capitalizing on strategic behavior.

2. Literature Review

Literature on pass pricing is scarce. First, we briefly review the literature on unlimited passes and then focus on several areas related to different aspects of limited passes.

Unlimited passes and a two-part tariff. Carbajo (1988) studied unlimited passes as a special case of a two-part tariff. Literature on two-part tariffs goes back to Oi (1971). Two-part tariffs cannot be used to analyze limited passes because, for a given customer in a particular selling horizon, an unlimited pass is a one-time purchase whereas a limited pass may result in multiple repeated purchases (renewals).

We consider dynamic pricing of a fixed number of credits. Certain aspects of this setup have been addressed separately in the marketing literature (mostly with static prices), but our model combines some unique features: (i) dynamic pricing, (ii) strategic credit utilization process, (iii) capacity constraint.

Advance purchase. The pass requires customers to prepay for future consumption. Since the payment and consumption times are separate, the pass can be viewed as a form of advance payment. In advance purchase, the customers only pay for the right of future consumption and that may not equal the actual consumption. The benefits of advance purchase are known in the literature. Shugan and Xie (2000) suggest that advance selling can mitigate a firm's information disadvantages by shifting some uncertainty to the customers. Xie and Shugan (2001) show that advance selling can generate higher profit when future customers are uncertain about their valuation. Our paper differs from these studies by its focus on exploring the benefits of advance selling of *multiple units* in the face of *strategic customers.*

Multi-unit pricing. The pass is related to multi-unit pricing because it is a package of identical products or services. In reality, passes almost always offer some quantity discounts (where the unit price decreases with the purchase quantity). The literature on quantity discounts is very rich in economics and marketing (Wilson, 1993). It is known that quantity discounts can price-discriminate customers with different consumption volumes to improve the revenue (Dolan, 1987). However, the majority of quantity discount models are static. Recently, Levin et al. (2014) explored the quantity discounts in dynamic pricing without considering the customer's choice among different quantities. However, in the presence of passes, customers can indeed choose between an individual unit or a package of multiple units. Existing literature on multi-unit pricing with customer choice does not distinguish the quantity and quality aspects of the product (Maskin and Riley, 1984, Tirole, 1988, Stole, 2007). A common assumption is that the utility of multiple units depends only on the number of units, so a package of multiple units is treated as a distinct unit with a higher quality. This assumption is questionable in the presence of strategic customers facing expiration time and dynamic prices, because the perceived utility of multiple credits depends on the strategic consumption process, which can be highly dynamic. For example, the utility of credits will diminish as one approaches the expiration time or the capacity is depleted. The utility of credits also depends on future prices that are time-dependent. All of these temporal effects have not been captured by the existing literature.

The sale of a package of multiple units can create a temporary drop in future demand, because the customer who bought the package has little incentive to buy more before consuming these multiple units. A similar phenomenon also occurs after a promotion (known as a post-promotion dip), because some customers purchase multiple units during the promotion and stockpile them for future consumption (Macé and Neslin, 2004, Su, 2010). Such temporal cannibalization is another difference between pass pricing and existing multi-unit pricing models.

Advance purchase of multiple units (bucket pricing). For the limited pass, the issuer specifies a maximum allowance for consumption (quota) and the customers prepay a flat fee for the quota. Hence, the pass is similar to a combination of an advance purchase and a multi-unit package, also known as bucket pricing, see Sun et al. (2006). Bucket pricing is commonly practised in the telecommunication and software industries. It differs from quantity discount in that the price is not increasing in the consumption volume. The customers pay a flat fee for any amount of consumption below the quota. Under bucket pricing, the customers often do not use all the amount that they have paid for. This may be due to uncertainty about future consumption at the time of purchase (Sun et al., 2006), or because some customers feel that consuming only a part of the package provides sufficient value (since the package offers a quantity discount). Some customers may even fail or forget to use up the quota before the expiration date. The resulting under-utilization allows the issuer to oversell their inventory and capitalize on the unused items (also known as breakage).

The literature on bucket pricing is dominated by empirical studies examining customer choice among different static price/quota combinations. Sun et al. (2006) studied customer choice among multiple plans in the on-line DVD rental industry. They found that customers often overpay for their actual consumption due to uncertainty about future consumption. Further, their empirical findings suggest that the long-term perception of cost and capacity affects customer purchasing behavior (which supports our problem setup). More recently, Schlereth and Skiera (2012) developed a choice model for different services. However, these studies consider customer choice rather than the optimal pricing decisions. There is also a major difference between pass pricing and the bucket pricing: the time horizon for bucket pricing programs is typically short so that most customers cannot reach the quota before the expiration time. In contrast, the pass pricing problem typically includes a relatively long horizon that allows pass holders to spend all their credits and consider repeat purchases (or renewals).

The time of a repeat purchase depends on the consumption process that, in turn, is affected by the pricing policy. Empirical studies show that the consumption rate depends on the remaining balance as well as the time-to-expiration (Andrews et al., 2013). With sufficient remaining balance, customers tend to consume at a faster rate without worrying about renewing at high prices (Assuncao and Meyer, 1993). The consumption rate will decrease with consumption as a lower credit balance is perceived more valuable (Folkes et al., 1993). Pricing decisions should account for their own impact on this nonlinear consumption process.

Mixed bundle pricing. In the pass pricing problem, the issuer offers customers a choice between an individual item and a bundle of items (the pass). In this sense, the problem is related to mixed bundle pricing. However, pass offers a package of *identical* products or services, which is different from a bundle because the latter typically consists of several *different* products. The bundle pricing literature focuses on the correlation in demand for different products (Venkatesh and Mahajan, 1993, Chu et al., 2011), which is different from our focus on the pass purchase and utilization process.

Capitalizing on strategic behavior. The phenomenon of capitalization on strategic behavior is related to a wider topic of mitigation, which is surveyed, e.g., by Aviv et al. (2009). Some of the most effective mitigation approaches include strategic rationing to induce early purchases (Liu and van Ryzin (2008)) and a quick replenishment of the product stock (Cachon and Swinney (2009)). Some studies also demonstrated that the firm can benefit from strategic behavior. For example, in a market where customers are heterogeneous both in valuation and in the level of patience, Su (2007) showed that rationing of sales, in combination with pricing, can be used by the firm to create a risk of shortage and hence encourage customers to buy early. When high-valuation customers are myopic but low-valuation ones are strategic, the firm can gain from their strategic behavior. Cho et al. (2009) show that strategic behavior can benefit both the firm and the customers. Bazhanov et al. (2014) study the quantity competition and show that, under some conditions, the firm's total profit can be non-monotone in the level of strategic behavior. A recent empirical study by Li et al. (2014) also identified the capitalization effect in airlines. However, most of the prior literature emphasizes the role of capacity or the availability of the product. In contrast, we find that introducing the pass is a promising new approach to harness strategic behavior without the need to limit supply. Instead of creating a stock-out risk, the pass offers a prepaid quantity discount. In order to get the discount, the customers have to purchase early (prepay) and buy a larger quantity. So the pass boosts the demand and revenue through a mechanism different from the existing literature.

3. The Model

The problem takes place over a finite horizon $[0, \tau]$ with a continuous time index $t \in [0, \tau]$. A total capacity of $\bar{z}$ items can be sold over this horizon. Items can be sold individually as well as through passes. One decision is to determine the regular price, i.e., the price of one item. Since passes can also be sold at different price levels, another decision is the price of the pass. We use f_t and p_t to denote, respectively, the regular price and the pass price at time t. The control process is a combination of both prices $c_t = (f_t, p_t)$. Each pass initially has $\bar{k} + 1$ credits. Since, in practice, pass utilization typically commences very quickly after the purchase, we assume that the first credit is consumed immediately and the pass holder has $\bar{k}$ credits to use in the future. We also suppose that all passes expire at the end of the horizon τ which is consistent with a practice of seasonal passes. In a general setting, this assumption results in an approximation that applies if the pass validity period is sufficiently long for pass holders to spend all credits.

When the setting is deterministic from the issuer's point of view, the control problem can use policies in an open-loop form. That is, the issuer optimizes the policy at the very beginning and uses it throughout the time horizon. In such settings, it is also convenient to use the fluid model by treating the capacity and market demand as continuous quantities. In the fluid model, sales occur continuously and come from the following sources: (1) an item may be sold without the use of passes, (2) a sale may occur with a new pass purchase, and (3) a sale may occur using a credit from one of the existing pass holders.

We let the intensity of quote requests originating from the entire population (normalized to one) be a constant λ. A quote request that originates from a non-pass-holder may or may not be converted into a pass purchase. A pass holder remains a potential source of additional purchases later when she spends all the credits on the pass. Each existing pass with unused credits generates utilization requests with the same intensity as a customer without a pass. Customers are inherently forward-looking and their choice probabilities depend on their perceptions of the future utility of consumption with and without passes. Therefore, this utility must be included into the description of the system state.

The real-life pass purchase and utilization process is quite complex. Along with the pricing policy and the pass expiration date, one of the most important drivers of customer behavior is the number of remaining credits. Leaving the treatment of a general pass expiration date for future work, we can assume that the booking behavior of customers depends only on the pricing policy and the number of remaining credits. Moreover, the dependence on the number of remaining credits $k = 0, \dots, \bar{k}$ is essentially captured by the utility of these credits, denoted as U_{kt}. Consequently, we can specify choice probabilities as functions of k, c_t and the vector of utilities U_t. Given c_t that specifies the regular and pass pricing decisions at time t, and the current vector of future utilities U_t, a customer without a pass who has generated a quote request at time t purchases a pass and consumes one item with probability $\pi^p_{0t}(c_t, U_t)$ putting him into a state with $\bar{k}$ credits. Similarly, this customer decides to buy a single item with probability $\pi^s_{0t}(c_t, U_t)$, but such a purchase keeps the customer in the state with zero credits. A customer with k credits on a pass arriving at time t decides to use his pass to consume an item with probability $\pi^p_{kt}(c_t, U_t)$ moving him to a state with $k-1$ credits. As a result of this specification, the number of credits held by a particular customer evolves according to a recurrent continuous-time Markov jump process. We let the probability that a particular customer has $k = 0, \dots, \bar{k}$ credits be w_{kt}. Because of our normalization of the market size to one, w_{kt} represents the fraction of customers with k credits at time t, and the vector $w_t = (w_{0t}, \dots, w_{\bar{k}t})$ represents the distribution of the number of credits in the customer population.

Overall quote request intensity λ is split among customers with different k in proportion to w_{kt}. The resulting intensities of single and pass purchases by customers without a pass are $\lambda w_{0t}\pi^s_{0t}(c_t, U_t)$ and $\lambda w_{0t}\pi^p_{0t}(c_t, U_t)$, respectively. The quantity $\lambda\pi^p_{0t}(c_t, U_t)$ plays the role of intensity of transitions from state 0 to state $\bar{k}$ in the pass purchase and credit utilization Markov process. Similarly, $\lambda\pi^p_{kt}(c_t, U_t)$ is the intensity of transitions from state k to $k-1$, which is also the utilization intensity by customers with k credits. Along with affecting w_t, pass utilization reduces the available capacity. The issuer oversees a large number of purchase options with their aggregated capacity sold over the horizon $[0, \tau]$. We use z_t to denote the remaining aggregate capacity at time t. We also let $\bar{z}$ represent the initial capacity and assume that overselling is not allowed (that is, z_t remains nonnegative).

The objective of the issuer is to maximize the revenues from the available capacity by controlling regular and pass prices subject to dynamics of the state variables that include the remaining capacity, the breakdown of customers according to the number of remaining credits on their passes, and customer perceptions of future utility. Next, we describe state dynamics and specify the choice models.

3.1. **Choice model.** We adapt the classical discrete choice model based on linear random utility (as described in Anderson et al. (1992)) to the context of strategic customer behavior. Following the approach of Levin et al. (2009), who used a certainty equivalent of future purchase to access the value of a no-purchase option, we generalize this notion to a certainty equivalent of the value of future consumption *given* a particular number of pass credits. One more difference from Levin et al. (2009) is that the current model is continuous-time. Assuming risk-neutral customers, the generalized certainty equivalent is equal to the utility U_{kt} of future consumption given k credits at time t. We will show that this utility can be computed from the Markov pass purchase and credit utilization process to be described in §3.2. For the purposes of the current section, we treat it as given.

In addition to making individual purchases at the respective single-item prices, the customers can purchase a pass and acquire items with credits. It is conceivable that someone with unused pass credits can still consider buying individual items at regular prices or, possibly, even buying a new pass prior to existing pass' expiration. However, both of these possibilities are unlikely because of the "sunk-cost" bias: pass holders are compelled to use credits they have already paid for to avoid feeling that they have wasted their money. Moreover, pass holders may feel the urgency to spend the credits before the expiration date (since they generally cannot resell or transfer the credits by terms and conditions of passes) or, in some instances, passes may provide more convenient ways to access the service. Thus, we assume that existing pass holders focus on using the remaining pass credits and do not consider purchases at regular price or new pass purchases until all their credits are finished. This greatly simplifies the description of the system state and customer utility.

Consider a customer who currently does not have a pass and is facing pricing policies c_t at time t. The value of a no-purchase option for this customer is $u^n_{0t} = U_{0t} + \epsilon^n_{0t}$, where U_{0t} is the expected value of this option (equal to the expected value of future consumption) and ϵ^n_{0t} is a random component of utility (with mean zero). Similarly, the value of purchasing an item at a regular price without the pass is $u^s_{0t} = a - f_t + \beta U_{0t} + \epsilon^s_t$, where a is the average valuation for consumption of one item and ϵ^s_t is a random utility component. We include a post-immediate consumption discount factor β that measures a subjective reduction in the expected future utility given that the current purchase takes place. The value of β represents the subjective relative reduction in utility due to temporarily reduced needs right after the current purchase. A more refined but also more complex alternative to a discount β would be an explicit treatment of the post-consumption states in the purchase and pass utilization process. On the balance of complexity versus simplicity, we opt for simplicity since a more complex model would not address substantial additional effects in this setting. In the present model, the expected value of the future consumption βU_{0t} is additive with the expected surplus of one item $a - f_t$. The resulting utility is compared to the value of a no-purchase option and the value of buying a pass. The value of pass purchase (in conjunction with consuming an item using the first credit on the pass) is $u^p_{0t} = a - p_t + \beta U_{\bar{k}t} + \epsilon^p_t$. The choice is to buy at regular price if $u^s_{0t} = \max\left\{u^n_{0t}, u^s_{0t}, u^p_{0t}\right\}$ and to buy the pass if $u^p_{0t} = \max\left\{u^n_{0t}, u^s_{0t}, u^p_{0t}\right\}$. The choice probability can be obtained in closed form for some distributions of the random utility component. A widely used form is the multinomial logit (MNL) model in which ϵ^n_{0t}, ϵ^s_t and ϵ^p_t are Gumbel-distributed. The random components of utility may be dependent for decisions to consume on a pass or as a single purchase. Thus, we assume the choice probabilities are determined by the nested logit (NL) model (Williams, 1977, McFadden, 1978):

$$\pi^p_{0t}(c_t, U_t) = \frac{\exp\left(\frac{a - p_t + \beta U_{\bar{k}t} - U_{0t}}{\mu\gamma}\right)\left[\exp\left(\frac{a - p_t + \beta U_{\bar{k}t} - U_{0t}}{\mu\gamma}\right) + \exp\left(\frac{a - f_t - (1-\beta)U_{0t}}{\mu\gamma}\right)\right]^{\gamma-1}}{1 + \left[\exp\left(\frac{a - p_t + \beta U_{\bar{k}t} - U_{0t}}{\mu\gamma}\right) + \exp\left(\frac{a - f_t - (1-\beta)U_{0t}}{\mu\gamma}\right)\right]^{\gamma}}, \quad (1)$$

$$\pi^s_{0t}(c_t, U_t) = \frac{\exp\left(\frac{a - f_t - (1-\beta)U_{0t}}{\mu\gamma}\right)\left[\exp\left(\frac{a - p_t + \beta U_{\bar{k}t} - U_{0t}}{\mu\gamma}\right) + \exp\left(\frac{a - f_t - (1-\beta)U_{0t}}{\mu\gamma}\right)\right]^{\gamma-1}}{1 + \left[\exp\left(\frac{a - p_t + \beta U_{\bar{k}t} - U_{0t}}{\mu\gamma}\right) + \exp\left(\frac{a - f_t - (1-\beta)U_{0t}}{\mu\gamma}\right)\right]^{\gamma}}, \quad (2)$$

where μ is a positive scaling parameter describing the variance of the random component of utility, and $\gamma \in (0, 1]$ is a coefficient that describes the customer's perceived similarity of consuming an item at a regular price and with a pass. The nested logit model can capture the correlation in customer's utilities of these two options. When $\gamma = 1$, the NL model reduces to MNL model. In the extreme case of $\gamma \to 0$, the utility of consumption of an individual item bought at a regular price and an item secured with the first credit on a pass is identical, but, in reality, it may not be perfectly correlated. Thus, parameter γ permits the model to capture settings that range from perfect correlation to independence in utility.

TABLE 1. List of parameters

Parameter	Range	Interpretation
λ	$(0,\infty)$	Quote request intensity
τ	$(0,\infty)$	Sales horizon length
$\bar{z}$	$(0,\infty)$	Total capacity
a	$(0,\infty)$	Avg. valuation of regular purchase
a^P	$(0,\infty)$	Avg. valuation of credit utilization
μ	$(0,\infty)$	Scale parameter for unobserved utility
ρ	$[0,\infty]$	Strategicity parameter
β	$[0,1]$	Subjective discount factor
γ	$(0,1]$	Degree of independence in unobserved utility

According to the assumption on the valid pass holders, their behavior is limited to choosing whether to use the remaining credits. The value of not using the credit k is $u^n_{kt} = U_{kt} + \epsilon^n_{kt}$. Using the credit k has the value $u^p_{kt} = a^P + \beta U_{(k-1)t} + \epsilon^p_t, \quad k = 1,\ldots,\bar{k}$, where a^P is the average valuation of consumption with a pass. Although a^P is allowed to take any positive value, we would expect $a^P \geq a$ in most cases due to the "sunk-cost" bias. The decision is to use the credit k if $u^p_{kt} = \max\{u^n_{kt}, u^p_{kt}\}$. The corresponding choice probability takes the form

$$\pi^p_{kt}(c_t, U_t) = \pi^p_{kt}(U_t) \quad = \quad \frac{\exp\left(\frac{a^P+\beta U_{(k-1)t}-U_{kt}}{\mu}\right)}{1+\exp\left(\frac{a^P+\beta U_{(k-1)t}-U_{kt}}{\mu}\right)}, \quad k = 1,\ldots,\bar{k}. \tag{3}$$

In the next section, we state the dynamic equations guiding the evolution of the available capacity, customer distribution across the number of credits, and the expected utility. For the latter, the critical remaining behavioral parameter is the discount factor $\rho > 0$ that describes the degree of foresight or the level of strategic behavior of the customers. A smaller value of ρ represents more strategic customers. The case of $\rho = 0$ represents fully strategic customers, and $\rho = \infty$ represents myopic customers. This parameter functions as a standard discount factor in the dynamic equations for utility and corresponds to fully rational aspect of strategic behavior. This is in contrast with the factor β which may also represent additional "irrational" aspects of strategic behavior. For example, Ellsberg (1961) showed that a majority of decision makers avoid alternatives with increased uncertainty to the point of ceasing to be the rational utility maximizers (Ellsberg's paradox). We will use β to potentially capture the aversion to additional customer uncertainty resulting from a subjective delay in returning to an active consumption state at a later point.

3.2. **Pass purchase and credit utilization process.** We first describe the dynamics and the boundary conditions for remaining capacity and the distribution of customers according to the number of remaining credits, which are the state variables of the model. The right-hand side (RHS) of the ordinary differential equation for each state variable combines purchase and utilization intensities from all sources that affect it. Remaining aggregate capacity is subject to the following differential equation

$$\dot{z}_t = -\lambda\Big\{w_{0t}\big[\pi^s_{0t}(c_t, U_t) + \pi^p_{0t}(c_t, U_t)\big] + \sum_{k=1}^{\bar{k}} w_{kt}\pi^p_{kt}(U_t)\Big\}, \tag{4}$$

with the boundary conditions $z_0 = \bar{z}$ and $z_\tau \geq 0$ representing the initial capacity and the no-oversales restriction, respectively. The distribution of customers according to the number of remaining credits is subject to

$$\dot{w}_{kt} \quad = \quad \lambda w_{(k+1)t}\pi^p_{(k+1)t}(U_t) - \lambda w_{kt}\pi^p_{kt}(c_t, U_t), \;\; k = 0,\ldots,\bar{k}-1, \tag{5}$$

$$\dot{w}_{\bar{k}t} \quad = \quad \lambda w_{0t}\pi^p_{0t}(c_t, U_t) - \lambda w_{\bar{k}t}\pi^p_{\bar{k}t}(U_t), \tag{6}$$

with boundary conditions $w_{k0} = 0$, for $k = 1, \ldots, \bar{k}$, and $w_{00} = 1$, which suggest that the population starts without the pass at all. Assuming that c_t and U_t are given, equations (6) correspond to standard evolution equations of a distribution vector of a continuous-time Markov jump process.

Next, define the marginal utility of credit k as $\Delta U_{kt} \triangleq U_{kt} - \beta U_{(k-1)t}$, $k = 1, \ldots, \bar{k}$. The marginal utility vector is denoted by $\Delta U_t = (U_{0t}, \Delta U_{1t}, \ldots, \Delta U_{\bar{k}t})$. The probabilities $\pi^p_{0t}(c_t, U_t)$, $\pi^s_{0t}(c_t, U_t)$, $\pi^p_{kt}(U_t)$, $k = 1, \ldots, \bar{k}$ depend on the utility only through the utility vector ΔU_t. Therefore, with a slight abuse of notation, we write them as $\pi^p_{0t}(c_t, \Delta U_t)$, $\pi^s_{0t}(c_t, \Delta U_t)$, $\pi^p_{kt}(\Delta U_t)$. The dynamics of the marginal utility vector are described in the following lemma.

Lemma 1. *The marginal utility vector ΔU_t satisfies the following differential equations*

$$\dot{U}_{0t} = \lambda\mu \ln(1 - \pi^p_{0t} - \pi^s_{0t}) + \rho U_{0t}, \tag{7}$$

$$\Delta\dot{U}_{1t} = \lambda\mu\big\{ \ln(1 - \pi^p_{1t}(\Delta U_t)) - \beta \ln\left[1 - \pi^p_{0t}(c_t, \Delta U_t) - \pi^s_{0t}(c_t, \Delta U_t)\right]\big\} + \rho\Delta U_{1t}, \tag{8}$$

$$\Delta\dot{U}_{kt} = \lambda\mu\big\{ \ln[1 - \pi^p_{kt}(\Delta U_t)] - \beta \ln[1 - \pi^p_{(k-1)t}(\Delta U_t)]\big\} + \rho\Delta U_{kt},\ k = 2, \ldots, \bar{k}, \tag{9}$$

with boundary conditions $U_{0\tau} = \Delta U_{k\tau} = 0$, $k = 1, \ldots, \bar{k}$. Furthermore, $U_{0t} \geq 0$ and $\Delta U_{kt} \geq 0$, $k = 1, \ldots, \bar{k}$.

The boundary conditions state that the utility of any unused credits on their expiration date is zero.

3.3. Objective function.

The objective is to maximize the total revenue over a finite horizon, i.e.,

$$\max_{c_t} \int_0^\tau R(c_t)dt \triangleq \max_{c_t} \int_0^\tau \lambda w_{0t}\Big[p_t \pi^p_{0t}(c_t, \Delta U_t) + f_t \pi^s_{0t}(c_t, \Delta U_t)\Big]dt \tag{10}$$

subject to state equations (4)-(9) and the corresponding boundary conditions.

4. Optimal Control Formulation

We apply the Pontryagin's maximum principle to derive the necessary conditions for the optimal pricing policy (see Sethi and Thompson (2000)). Since the revenue rate $R(c_t)$ incorporates the nested logit probabilities of customer choices, it is not jointly concave in the prices $c_t = \{f_t, p_t\}$ (see Gallego and Wang (2014)). However, it is concave in the choice probabilities (see Li and Huh (2011)). Thus, we reformulate the problem by using the choice probabilities $u_t = \{\pi^s_{0t}, \pi^p_{0t}\}$ as the control variables, and define the corresponding convex control set as $\mathbb{U} = \big\{(\pi^p_{0t}, \pi^s_{0t}) \in (0,1)^2 : \pi^p_{0t} + \pi^s_{0t} < 1\big\}$. We also let $x_t = (w_t, \Delta U_t) \in \mathbb{X}$ denote the vector of state variables excluding z_t, where $\mathbb{X}$ is the state space. The revenue rate in new variables is $R(u_t)$.

The prices are now functions of the new control and state variables:

$$f_t = a + \mu\gamma\Big[\ln(1 - \pi^p_{0t} - \pi^s_{0t}) - \ln \pi^s_{0t}\Big] + \mu(1-\gamma)\Big[\ln(1 - \pi^p_{0t} - \pi^s_{0t}) - \ln(\pi^p_{0t} + \pi^s_{0t})\Big] - (1-\beta)U_{0t},$$

$$p_t = a + \mu\gamma\Big[\ln(1 - \pi^p_{0t} - \pi^s_{0t}) - \ln \pi^p_{0t}\Big] + \mu(1-\gamma)\Big[\ln(1 - \pi^p_{0t} - \pi^s_{0t}) - \ln(\pi^p_{0t} + \pi^s_{0t})\Big]$$
$$+ \sum_{k=1}^{\bar{k}} \beta^{\bar{k}-k+1}\Delta U_{kt} - (1 - \beta^{\bar{k}+1})U_{0t}$$

For brevity, we have dropped the arguments u_t and ΔU_t in the expressions of f_t, p_t. After introducing the new control variables, the state equations (4)-(9) remain unchanged except that the functions $\pi^p_{0t}(c_t, \Delta U_t)$ and $\pi^s_{0t}(c_t, \Delta U_t)$ are replaced by the control variables π^s_{0t}, π^p_{0t}.

According to the classical interpretation, the Hamiltonian function combines the direct instantaneous contribution to the objective with the indirect contributions of the state variables:

$$H \triangleq R(u_t) + H^z \eta_t^z + \sum_{k=0}^{\bar{k}} H_k^w \eta_{kt}^w + \sum_{k=0}^{\bar{k}} H_k^u \eta_{kt}^u,$$

where the adjoint variables η_t^z, η_{kt}^w and η_{kt}^u for the capacity, population distribution and utility, respectively, are interpreted as shadow prices. The functions H^z, H_k^w, and H_k^u, $k = 0, \dots, \bar{k}$, are the RHS of the state dynamics equations for, respectively, capacity (4), population (5)-(6) and utility (7)-(9). The vector $\eta_t = (\eta_t^z, \eta_{kt}^w, \eta_{kt}^u)$ denotes all the adjoint variables at time t.

The adjoint variable η_t^z, the shadow price of capacity, can be also viewed as the penalty of violating the capacity constraint. It satisfies the following adjoint equation and transversality condition

$$\dot{\eta}_t^z = -\frac{\partial H}{\partial z_t} = 0, \qquad \eta_t^z = \begin{cases} \eta^z > 0, & z_\tau = 0, \\ 0, & z_\tau > 0, \end{cases} \tag{11}$$

for all $t \in [0, \tau]$. In general, transversality conditions determine the shadow prices at the endpoints of the state trajectory. In this case, the adjoint equation implies that η_t^z stays constant and we denote it by η^z. The value of η^z depends on whether the capacity should be cleared under the optimal pricing policy. When the capacity should be cleared, i.e., $z_\tau = 0$, which we call the case of *limited capacity*, we have $\eta^z > 0$ or a positive shadow price. Otherwise, if some capacity remains unsold at the end of sales horizon, i.e., $z_\tau > 0$, which we call the case of *ample capacity*, $\eta^z = 0$ (zero shadow price).

The adjoint variables η_{kt}^w satisfy the following adjoint equations:

$$\begin{aligned} \dot{\eta}_{0t}^w &= -\frac{\partial H}{\partial w_{0t}} = -\lambda \Big\{ f_t \pi_{0t}^s + p_t \pi_{0t}^p - (\pi_{0t}^s + \pi_{0t}^p)\eta^z - \pi_{0t}^p(\eta_{0t}^w - \eta_{\bar{k}t}^w) \Big\}, \\ \dot{\eta}_{kt}^w &= -\frac{\partial H}{\partial w_{kt}} = \lambda \pi_{kt}^p [\eta^z + (\eta_{kt}^w - \eta_{(k-1)t}^w)], \quad k = 1, \dots, \bar{k}. \end{aligned} \tag{12}$$

where the boundary condition $\eta_{k\tau}^w = 0, k = 0, \dots, \bar{k}$ is implied by the transversality (i.e., the population distribution has no indirect effect on the objective at the very end of the sales horizon). For the remaining sections, we introduce new variables $\Delta\eta_{kt}^w = \eta_{(k-1)t}^w - \eta_{kt}^w, k = 1, \dots, \bar{k}$ and $\Delta\eta_{0t}^w = \eta_{0t}^w - \eta_{\bar{k}t}^w$ in the adjoint equations. The variable $\Delta\eta_{kt}^w$ denotes the *marginal shadow price of credit* k. Summing all the marginal shadow prices gives $\Delta\eta_{0t}^w$, namely, $\Delta\eta_{0t}^w = \sum_{k=1}^{\bar{k}} \Delta\eta_{kt}^w$.
The adjoint variables η_{kt}^u satisfy the adjoint equations:

$$\begin{aligned} \dot{\eta}_{0t}^u &= -\frac{\partial H}{\partial U_{0t}} = \lambda w_{0t} \big[(1 - \beta^{\bar{k}+1})\pi_{0t}^p + (1 - \beta)\pi_{0t}^s \big] - \rho \eta_{0t}^u, \\ \dot{\eta}_{kt}^u &= -\frac{\partial H}{\partial \Delta U_{kt}} = -\lambda w_{0t} \pi_{0t}^p \beta^{\bar{k}-k+1} + \frac{\lambda}{\mu} w_{kt} \pi_{kt}^p (1 - \pi_{kt}^p)(\Delta\eta_{kt}^w - \eta^z) \\ &\qquad + \lambda \pi_{kt}^p (\beta \eta_{(k+1)t}^u - \eta_{kt}^u) - \rho \eta_{kt}^u, \qquad k = 1, \dots, \bar{k} - 1, \\ \dot{\eta}_{\bar{k}t}^u &= -\frac{\partial H}{\partial \Delta U_{\bar{k}t}} = -\lambda w_{0t} \pi_{0t}^p \beta + \frac{\lambda}{\mu} w_{\bar{k}t} \pi_{\bar{k}t}^p (1 - \pi_{\bar{k}t}^p)(\Delta\eta_{\bar{k}t}^w - \eta^z) - \lambda \eta_{\bar{k}t}^u \pi_{\bar{k}t}^p - \rho \eta_{\bar{k}t}^u. \end{aligned} \tag{13}$$

with the boundary condition $\eta_{k0}^u = 0, k = 1, \dots, \bar{k}$. This transversality condition indicates that the utilities do not have an indirect contribution to the objective at the beginning of the sales season and all their contribution is captured by the revenue rate $R(u_t)$.

The following lemma suggests that the Hamiltonian evaluated along the optimal trajectory is jointly concave in the control variables.

Lemma 2. *Given any values of the state and adjoint variables with* $\beta\eta_{1t}^u - \eta_{0t}^u < w_{0t}$*, the Hamiltonian is concave in* $u_t = \{\pi_{0t}^s, \pi_{0t}^p\}$ *for all* $u_t \in \mathbb{U}$.

The concavity of the Hamiltonian is an appealing structure, since it simplifies the optimality conditions and is useful in structural results.

Lemma 3. *When $\beta\eta^u_{1t} - \eta^u_{0t} < w_{0t}$, the necessary and sufficient first order optimality conditions for the Hamiltonian are given by*

$$\frac{\partial H}{\partial \pi^s_{0t}} = 0 \Rightarrow w_{0t}\Big(f_t - \eta^z - \frac{\mu}{1-\pi^p_{0t}-\pi^s_{0t}}\Big) + (\beta\eta^u_{1t} - \eta^u_{0t})\frac{\mu}{1-\pi^p_{0t}-\pi^s_{0t}} = 0 \quad (14)$$

$$\frac{\partial H}{\partial \pi^p_{0t}} = 0 \Rightarrow w_{0t}\Big(p_t - \Delta\eta^w_{0t} - \eta^z - \frac{\mu}{1-\pi^p_{0t}-\pi^s_{0t}}\Big) + (\beta\eta^u_{1t} - \eta^u_{0t})\frac{\mu}{1-\pi^p_{0t}-\pi^s_{0t}} = 0 \quad (15)$$

The concavity and the first-order condition are only used for (1) the closed-form solution in two special cases and (2) turnpike revenue calculation and capitalization on strategic behavior. All results in Section 6 (except Lemma 5) do not rely on the first-order conditions.

Lemma 4 (Optimality Conditions)**.** *For any optimal controls (i.e., the purchase probability trajectory $\pi^{p\star}_{0t}, \pi^{s\star}_{0t}$) and the corresponding state trajectory (i.e., $z^\star_t, w^\star_t, \Delta U^\star_t$) satisfying the state equations* (4)-(9) *and their boundary conditions, there exists an adjoint trajectory (i.e., $\eta^{z\star}, \eta^{w\star}_{kt}, \eta^{u\star}_{kt}$) satisfying the adjoint equations* (11), (12), (13)*, together with their boundary conditions, such that, at each t, the values of purchase probabilities deliver the global maximum of the Hamiltonian. Moreover, when the utility adjoint variable $\beta\eta^u_{1t} - \eta^u_{0t} < w_{0t}$, the purchase probabilities satisfy the first order conditions* (14), (15).

The conditions stated in the lemma provide a characterization of the optimal purchase probabilities and the corresponding state and adjoint trajectories. The mathematical form of these conditions is a boundary-value problem for a system of differential and algebraic equations. The condition of the maximum of the Hamiltonian corresponds to the maximization of the combined direct and indirect instantaneous contribution to the objective at every point of the optimal purchase probability path. A straightforward structural observation made by subtracting (14) from (15) is

Proposition 1. *For the controls satisfying the first-order optimality conditions* (14)- (15) *for the Hamiltonian, the pass price can be decomposed into the regular price and a series of marginal shadow prices of credits, i.e.,*

$$p_t = f_t + \sum_{k=1}^{\bar{k}} \Delta\eta^w_{kt} \quad (16)$$

Further, the pass and regular ticket price are the same at the end of the horizon, i.e., $p_\tau = f_\tau$.

Proposition 1 provides a natural way of examining the structure of the price of a pass. As discussed before, a customer consumes a unit immediately after purchasing a pass, while the remaining $\bar{k}$ are reserved for future consumption. By purchasing a pass, the customer essentially prepays for these $(1 + \bar{k})$ uses: f_t is the amount paid for the first (or immediate) use, $\Delta\eta^w_{\bar{k}t}$ is the amount for the second use, ..., and $\Delta\eta^w_{1t}$ is for the last use with a credit. That is, the pass holder pays the regular price f_t for the immediate use and $\sum_{k=1}^{\bar{k}} \Delta\eta^w_{kt}$ for all future uses permitted by the pass. When the pass expires, the credits have zero value, so the pass price equals the regular price.

5. Turnpike Behavior

The optimality conditions given in §4 are a large system of nonlinear differential-algebraic equations that are prohibitive to solve analytically. However, this system exhibits a "turnpike" behavior, which leads to managerial insights and closed-form solutions in some cases.

An intuitive description of the turnpike behavior is that the optimal state and control trajectories stay near the steady state for most of the sales horizon if the horizon is sufficiently long. If the

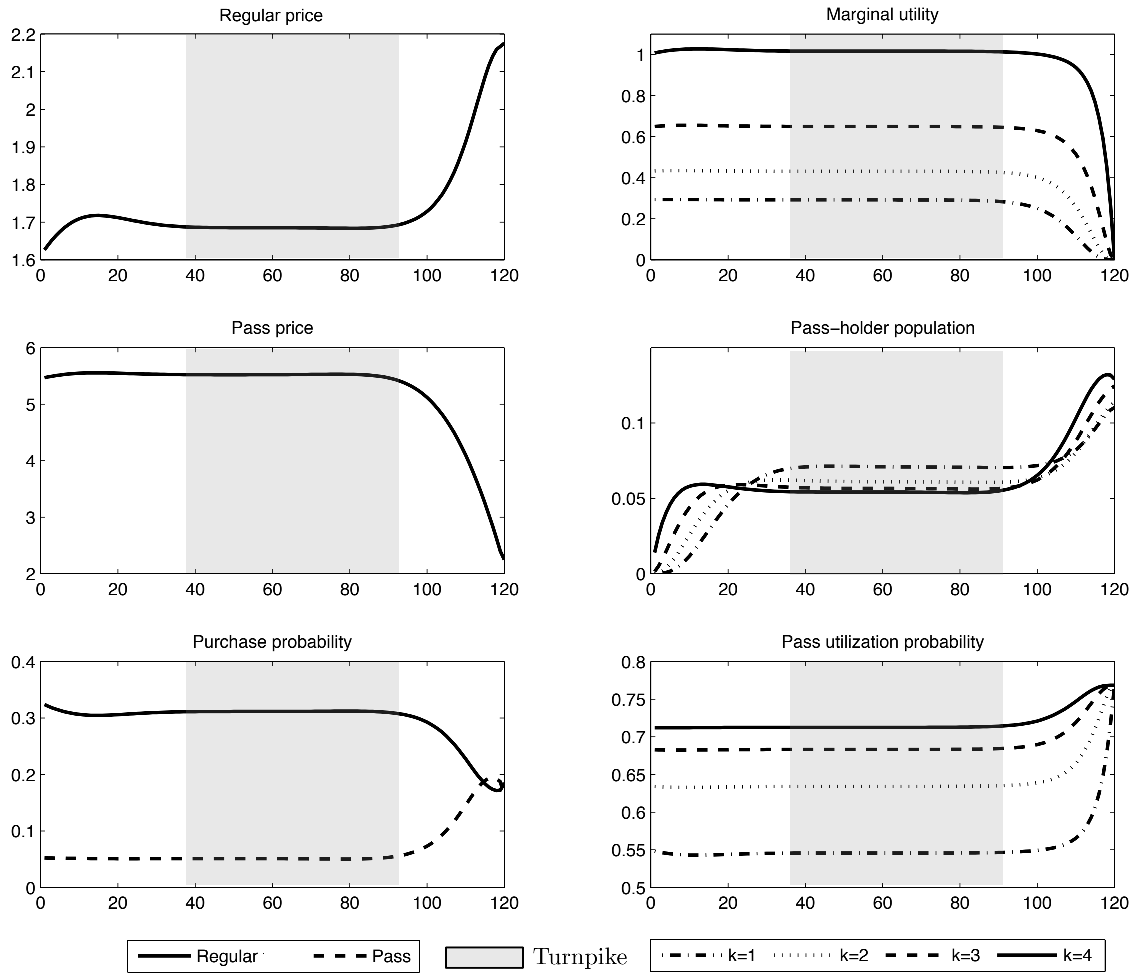

FIGURE 1. Illustration of turnpike properties in the state and control variables (for the case of $\bar{k} = 4$).

sales horizon is longer than a certain threshold (so that the steady state can be approached), the optimal trajectories converge to the turnpike regardless of the initial and terminal conditions. This behavior is very similar to real-world experience of driving along the major highway: if one wishes to drive from one city to another across a long distance, then she first gets on a highway, spends most of the time on it (similar to the steady state) and, finally, leaves the highway to reach the destination.

We first define the steady state equations that characterize the turnpike. Let $\bar{x} = (\bar{w}, \Delta\bar{U})$, $\bar{\eta} = (\eta^{z\star}, \bar{\eta}_k^w, \bar{\eta}_k^u)$ and $\bar{u} = (\bar{\pi}_0^p, \bar{\pi}_0^s)$ be the steady-state solution to the optimality conditions in Lemma 4. The steady-state solution is obtained by solving the nonlinear equations resulting from setting the derivatives in the state and adjoint equations to zero (while dropping the boundary conditions) and the first-order conditions for the Hamiltonian. The corresponding steady state prices are $\bar{f}, \bar{p}$, and the steady state credit utilization probabilities are $\bar{\pi}_k^p, k = 1, \ldots, \bar{k}$.

An example illustrating the temporal behavior of the optimal trajectories is shown in Figure 1. The optimal trajectories can be divided into three stages: an initial adjustment stage, a steady-state stage (highlighted in gray), and a terminal adjustment stage. The initial adjustment period is similar to the entrance ramp, which transfers the initial condition to the steady state. The second stage lies in the middle of the sales horizon and is the steady state (or turnpike, as shown

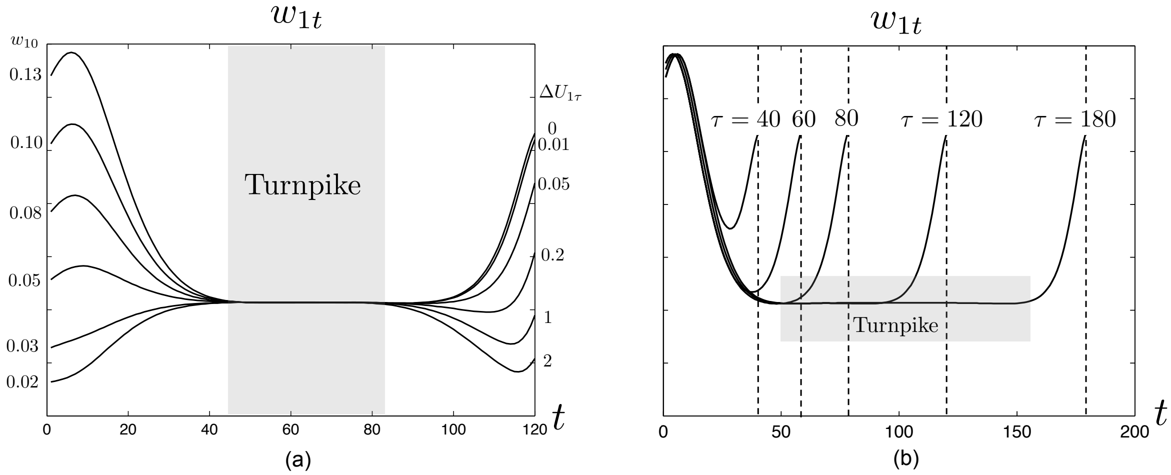

FIGURE 2. Optimal trajectories of w_{1t} ($\rho = 0.5$) illustrating that (a) the turnpike is independent of the boundary conditions. (b) the turnpike can only be reached when the sales horizon is long enough.

TABLE 2. The performance of fixed price approximation.

	Horizon	$\tau = 40$	$\tau = 60$	$\tau = 80$	$\tau = 120$	$\tau = 180$
$\rho = 0.5$	Optimal Rev.	5.17	8.30	16.11	31.76	47.41
	Approx. Rev.	4.73	7.88	15.71	31.36	47.01
	Approx. Error	**0.44**	**0.42**	**0.40**	**0.40**	**0.40**
	Approx. Error (%)	8.51%	5.06%	2.48%	1.26%	0.84%
$\rho = 5$	Optimal Rev.	4.91	8.02	15.80	31.38	46.97
	Approx. Rev.	4.69	7.81	15.60	31.18	46.77
	Approx. Error	**0.22**	**0.21**	**0.20**	**0.20**	**0.20**
	Approx. Error (%)	4.49%	2.62%	1.27%	0.64%	0.42%

in the shaded region), during which all state and control variables stay near constant levels. Upon reaching the terminal adjustment stage, state and control variables start to deviate from the steady state and make continuous adjustments to reach the terminal state (similar to the exit ramp).

In Figure 2 (a), we vary the initial credit distribution w_0 and the terminal marginal utility ΔU_τ to examine how the optimal trajectory of w_{1t} depends on the boundary conditions. We observe that the optimal trajectories associated with different boundary values converge to the same steady state and stay near it in the middle of the horizon. Figure 2 (b) illustrates how w_{1t} changes under varying length of the sales horizon τ. The steady state may not be reached when the horizon is short (for example, $\tau < 60$). But as long as the length of the sales horizon exceeds a certain level (for example, $\tau > 80$), the steady state can always be reached and the terminal adjustment stages appear very similar.

The existence of turnpike behavior naturally gives rise to the idea of using fixed steady-state prices to approximate the dynamic prices. Fixed price is appealing not only because it is simple to implement but asymptotically optimal in some settings (Gallego and van Ryzin, 1994). In Table 2, we compare the revenue from the optimal dynamic pricing policy with that from the fixed steady-state price, for different lengths of sales horizon and strategicity parameter. The parameters for this example are $(\bar{k}, \lambda, a, a^P, \mu, \gamma, \bar{z}, \beta) = (3, 0.25, 3, 3, 1, 0.8, 100, 1)$. The approximation error is initially decreasing in the horizon length τ when the horizon is short ($\tau < 80$) but remains stable when the horizon is long enough ($\tau > 80$). On the other hand, the revenue is increasing in the horizon length, so the percentage approximation error gradually diminishes as the sales horizon increases.

Table 2 also suggests that fixed price policy performs better when the customers are less strategic ($\rho = 5$). But the percentage error can be suppressed by a long horizon even when the strategic behavior is strong ($\rho = 0.5$). The good performance of fixed prices suggests that dynamic pricing can not significantly improve the revenue in the presence of passes. This is also consistence with practice in many settings where both passes and individual units are offered at flat rates.

Finally, it is natural to ask whether the sales horizons used in practice are long enough to make the turnpike reachable. Analysis of the pass purchase and utilization process can help to answer this question. At the initial adjustment stage, the pass purchase rate is greater than the pass utilization rate, so the number of pass holders increases. As more customers become pass holders, the pass purchase rate decreases but pass utilization rate increases, until they become balanced in the steady state. The steady state represents a realistic situation in which the pass holder can spend all the credits on the pass and then return to the market to make new purchases. A horizon that is not sufficiently long to allow a full utilization of the pass may ruin the issuer's reputation and long-term profit because the pass may not appeal to customers and/or may violate regulations.

5.1. **Myopic customers and fully strategic customers.** The closed-form expressions for the turnpike are quite unwieldy in general, but they become simple in the following two cases: myopic customers ($\rho = \infty$) and fully strategic customers ($\rho = 0$). Let $\bar{R}(\infty)$ and $\bar{R}(0)$ denote the turnpike revenue rate for myopic and fully strategic customers, respectively.

A small fraction of myopic customers may purchase the pass because they have high valuations. Since myopic customers have zero utilities for future consumption, they consume the credits at a constant probability $\pi^p_{kt} = \exp(a^P/\mu)/\big(1 + \exp(a^P/\mu)\big),\ k = 1, \ldots, \bar{k}$.

Proposition 2. ***(Myopic customers: two-part tariff)*** *If all customers are myopic ($\rho = \infty$), the turnpike regular price $\bar{f}$ is the root of the following equation*

$$\bar{f} - \eta^{z\star} = \mu\Big\{1 + \Big[\exp\Big(\frac{a - \bar{f}}{\mu\gamma}\Big) + \exp\Big(\frac{a - \bar{f} - \bar{k}(\bar{f} - \mu)(1 + e^{-a^P/\mu}) + \bar{k}\eta^z e^{-a^P/\mu}}{\mu\gamma}\Big)\Big]^\gamma\Big\}.$$

The credit price is $\Delta\bar{\eta}^w_k = \big[(\bar{f} - \mu)\big(1 + \exp(a^P/\mu)\big) - \eta^{z\star}\big]/\exp(a^P/\mu),\ k = 1, \ldots, \bar{k}$. The turnpike revenue rate is $\bar{R}(\infty) = \lambda(\bar{f} - \mu) - \lambda\big[\bar{w}_0(1 - \bar{\pi}^p_0 - \bar{\pi}^s_0) + (1 - \bar{w}_0)/\big(1 + \exp(a^P/\mu)\big)\big]\eta^{z\star}$.

Proposition 2 suggests that the pass should be priced according to a *two-part tariff* for myopic customers. The pass price consists of a lump-sum fee $\bar{f}$ and a per-unit charge $\Delta\bar{\eta}^w_k$, which remains constant for all $k = 1, \ldots, \bar{k}$.

We now consider fully strategic customers for the case of $\beta = 1$. In order to present the results, we first introduce a generalization of the golden ratio, which fully determines many important turnpike variables. The golden ratio, $(1 + \sqrt{5})/2 \approx 1.618$, is the positive solution to the quadratic equation $1 + x^{-1} = x$. We define the parameter $\alpha = \exp\Big((a^P - a)/(\mu\gamma)\Big)$ and generalize the golden ratio as

Definition 1. ***(Generalized golden ratio)*** *For a positive integer k, define ϕ^α_k as the real positive root of the equation $1 + x^{-k} = \alpha x$.*

The equation in Definition 1 has a unique positive root because its LHS is decreasing and its RHS is increasing in x. For the case of $\alpha = 1$ (equivalent to $a = a^P$), ϕ^α_1 reduces to the golden ratio $\phi^1_1 = (1 + \sqrt{5})/2 \approx 1.618$. We now give the closed-form solution to the turnpike.

Proposition 3. ***(Fully strategic customers: linear prices)*** *If $\beta = 1$ and customers are fully strategic ($\rho = 0$), the turnpike variables have the closed-form expressions: $\Delta\bar{U}_1 = \cdots = \Delta\bar{U}_{\bar{k}},\ \bar{\eta}^u_1 = \cdots = \bar{\eta}^u_{\bar{k}} = 0,\ \bar{f} = \Delta\bar{\eta}^w_1 = \Delta\bar{\eta}^w_2 = \cdots = \Delta\bar{\eta}^w_{\bar{k}},\ \bar{\pi}^p_0 + \bar{\pi}^s_0 = \bar{\pi}^p_1 = \bar{\pi}^p_2 = \cdots = \bar{\pi}^p_{\bar{k}},\ \bar{p} = (1 + \bar{k})\bar{f}, \bar{\pi}^p_0/\bar{\pi}^s_0 =$*

TABLE 3. Turnpike parameters determined by the generalized golden ratio ($\alpha = 1, \beta = 1, \rho = 0$)

Total credits	Pass holder population	Pass/Regular choice ratio
$\bar{k}$	$1 - \bar{w}_0$	$\bar{\pi}_0^p / \bar{\pi}_0^s$
	$\bar{k}/(1 + \bar{k} + (\phi_{\bar{k}}^1)^{\bar{k}})$	$\phi_{\bar{k}}^1 - 1$
1	0.2763	0.6180 (Golden ratio)
2	0.3885	0.4655
3	0.4525	0.3802
4	0.4950	0.3247
5	0.5259	0.2851

$\alpha\phi_{\bar{k}}^{\alpha} - 1, \bar{w}_1 = \cdots = \bar{w}_{\bar{k}} = \Big(1 + \bar{k} + (\phi_{\bar{k}}^{\alpha})^{\bar{k}}\Big)^{-1}$, *and*

$$\bar{f} - \eta^{z\star} = \mu\Big[1 + \exp\Big(\frac{a - \bar{f}}{\mu}\Big)(\phi_{\bar{k}}^{\alpha})^{\gamma}\Big].$$

The turnpike revenue rate is $\bar{R}(0) = \lambda(\bar{f} - \mu) - \lambda(1 - \bar{\pi}_0^p - \bar{\pi}_0^s)\eta^{z\star}$.

It can be easily verified that $\bar{f}$ is increasing in $\eta^{z\star}$. Some ratios determined by the generalized golden ratio (assuming $\alpha = 1$) are reported in Table 3. These ratios are fully determined by $\bar{k}$ (the total number of credits on the pass), which is known exactly. Therefore, they are robust to the parameter uncertainty.

6. Structural Results for the Turnpike

We now consider the turnpike price, demand, and revenue structure and their dependence on capacity and strategic behavior. While, in practice, passes almost always offer some discount compared to regular purchases, under what conditions does this occur? How does the discount influence the demand? Does the pass cannibalize the demand and revenue of individual sales in the future? Can pass sales generate higher revenue than the regular sales per customer? The goal of this section is to address these questions.

6.1. The demand structure.

The following proposition shows that passes can result in heterogeneous consumption probabilities, which are mediated by the strategic behavior.

Proposition 4 (Consumption monotonicity). *For strategic customers (i.e., $\rho < \infty$), the turnpike consumption probability is non-decreasing in the number of remaining credits k, i.e., $\bar{\pi}_0^s + \bar{\pi}_0^p \leqslant \bar{\pi}_1^p \leqslant \cdots \leqslant \bar{\pi}_{\bar{k}}^p$; the population fraction $\bar{w}_k$ is nonincreasing in k, i.e., $\bar{w}_0 \geqslant \bar{w}_1 \geqslant \bar{w}_2 \geqslant \cdots \geqslant \bar{w}_{\bar{k}}$. Moreover, all equalities can be simultaneously attained if and only if all customers are fully strategic (i.e., $\rho = 0$).*

This proposition suggests that passes *encourage* consumption in strategic customers: the customer with more credits is more likely to consume. The consumption probability is the highest when the pass is newly purchased, decreases with credit utilization, and reaches the lowest level when all credits are spent (in this case, the pass holder switches to regular purchase). This trend is consistent with the empirical observations of customer behaviors made by Andrews et al. (2013) in telecommunication and agrees with the scarcity theory in that fewer credits are perceived as more valuable. Proposition 4 also says that the steady-state distribution of the population over the number of credits is decreasing with zero credit being most likely. However, when customers are fully strategic ($\rho = 0$), the number of remaining credits has no impact on the consumption probability, and the population is uniformly distributed over the number of credits.

6.2. **Economic profit across customer states.** Customer state undergoes evolution over the lifetime of a pass, and Proposition 1 shows that a direct revenue p_t^* from each pass can be decomposed into indirect revenues associated with each use. In considering the contribution of each sale to the objective, we also need to account for the opportunity cost of capacity. This leads to the following definition:

Definition 2. *Define*

(1) *the* economic profit *as the price minus the optimal shadow price of capacity ($\eta^{z\star}$) and*
(2) *the* realized economic profit *as the economic profit times the* purchase *(or utilization) probability.*

When a customer uses a credit in a steady state, its realized indirect contribution to the objective should be the same regardless of the current number of credits. This is indeed true as indicated by the following proposition.

Proposition 5 (Constant realized economic profit)**.** *The turnpike realized economic profit is a constant. That is,* $(\bar{\pi}_0^s + \bar{\pi}_0^p)(\bar{f} - \eta^{z\star}) = \bar{\pi}_1^p(\Delta\bar{\eta}_1^w - \eta^{z\star}) = \cdots = \bar{\pi}_{\bar{k}}^p(\Delta\bar{\eta}_{\bar{k}}^w - \eta^{z\star})$.

The marginal shadow price $\Delta\bar{\eta}_k^w$ is the change in indirect contribution per unit of the state variable w_k (per earlier Hamiltonian interpretation) associated with a change in the number of credits. This is also an indirect revenue corresponding to credit k. The value $\eta^{z\star}$ is the opportunity cost of a unit of capacity. Combining them with the purchase probability $\bar{\pi}_k^p$, we obtain the realized economic profit that is constant for all k.

Proposition 5 suggests that every customer in the system generates the *same* realized economic profit in the turnpike, regardless of the number of credits remaining. It also implies the following monotone structure of the indirect contributions of credits to the revenue.

Corollary 1. $\bar{f}(\bar{\pi}_0^s+\bar{\pi}_0^p) \leqslant \bar{\pi}_{(k-1)}^p\Delta\bar{\eta}_{(k-1)}^w \leqslant \bar{\pi}_k^p\Delta\bar{\eta}_k^w,\ , k = 2, \ldots, \bar{k}$, *where the equalities are attained when all customers are fully strategic ($\rho = 0$) or the capacity is ample ($\eta^{z\star} = 0$).*

When the capacity is ample *or* when all customers are fully strategic, Corollary 1 suggests that every customer has the same indirect contribution to the revenue in the turnpike, regardless of the number of remaining credits. However, in industries practicing revenue management, one can anticipate that capacity is limited and customers are not fully strategic, so the inequalities should hold in most cases.

6.3. **The price structure.** Next, we study the relation between the turnpike regular price $\bar{f}$ and the marginal shadow prices $\Delta\bar{\eta}_k^w$ in the following

Corollary 2. ***(Nonlinear prices)*** *For strategic customers (i.e., $\rho < \infty$), the regular price and the marginal shadow prices of credits are ordered as $\bar{f} \geqslant \Delta\bar{\eta}_1^w \geqslant \cdots \geqslant \Delta\bar{\eta}_{\bar{k}}^w$. Furthermore, all equalities are simultaneously attained if and only if the customers are fully strategic (i.e., $\rho = 0$).*

This corollary suggests diminishing marginal prices for the credits. This nonlinear pricing structure resembles the quantity discount. However, the issuer is better off not offering any discount in the face of fully strategic customers. In this extreme case, the pass price is proportional to the number of credits on the pass, giving rise to a linear pricing structure, as shown in Proposition 3. When customers are myopic, then the pass should be priced as a two-part tariff by Proposition 2. In Figure 3, we present illustrations of the nonlinear pricing structure. The parameters are $(\bar{k}, \tau, \lambda, a, a^P, \mu, \gamma, \bar{z}, \beta) = (4, 120, 0.3, 1, 1.2, 2, 0.5, 100, 1)$. We observe that the nonlinear pricing structures (for partially strategic customers with $0 < \rho < \infty$) are bounded by the linear pricing and the two-part tariff. The non-linearity decreases when the customers become more strategic (as ρ decreases), which suggests that the issuer should offer smaller discounts to more strategic customers.

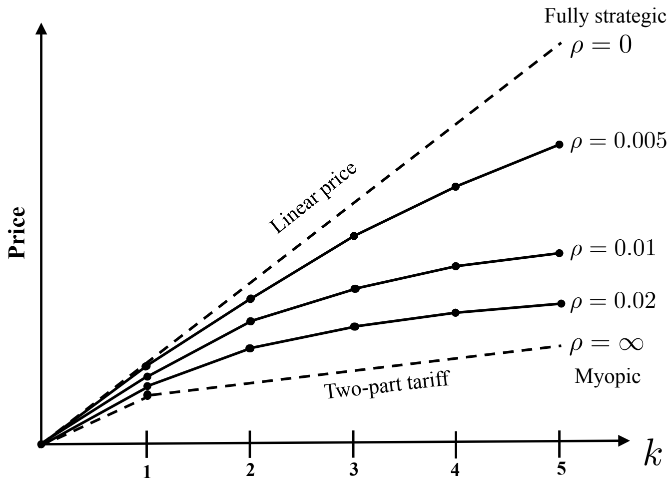

FIGURE 3. The nonlinear structure of pass price

6.4. **The revenue structure.** Next, we examine how a pass credit differs from a regular sale in revenue rate. Does the credit simply shift future revenue from regular sales to the present, or can it generate more revenue and when does this occur?

Corollary 2 suggests that passes can offer discounts for future credit-based consumption. Moreover, by Proposition 4, the credit-based consumption probabilities can be higher than the immediate consumption probabilities. The higher credit utilization probabilities result in more frequent pass purchases, leading to an increase in pass demand. The following proposition sheds some light on the net result of these two opposing effects.

Proposition 6. *(**Composition of revenue**) The turnpike revenue rate can be decomposed as follows*

$$\bar{R} = \underbrace{\lambda \bar{w}_0 \bar{f}(\bar{\pi}_0^s + \bar{\pi}_0^p)}_{\textit{immediate consumption}} + \underbrace{\lambda \sum_{k=1}^{\bar{k}} \bar{w}_k \Delta \bar{\eta}_k^w \bar{\pi}_k^p}_{\textit{credit}} \tag{17}$$

$$= \underbrace{\lambda \bar{w}_0 \bar{f}(\bar{\pi}_0^s + \bar{\pi}_0^p)}_{\textit{immediate consumption}} + \underbrace{\lambda \sum_{k=1}^{\bar{k}} \bar{w}_k \Big(\bar{f}(\bar{\pi}_0^s + \bar{\pi}_0^p) + \big[\bar{\pi}_k^p - (\bar{\pi}_0^s + \bar{\pi}_0^p)\big] \eta^{z\star} \Big)}_{\textit{credit}} \tag{18}$$

$$= \underbrace{\lambda \bar{w}_0 \bar{f}(\bar{\pi}_0^s + \bar{\pi}_0^p)}_{\textit{immediate consumption}} + \underbrace{\lambda \sum_{k=1}^{\bar{k}} \bar{w}_k \bar{f}(\bar{\pi}_0^s + \bar{\pi}_0^p)}_{\textit{base revenue rate of credit}} + \underbrace{\lambda \sum_{k=1}^{\bar{k}} \bar{w}_k \big[\bar{\pi}_k^p - (\bar{\pi}_0^s + \bar{\pi}_0^p)\big] \eta^{z\star}}_{\textit{capacity-scarcity surcharge}} \tag{19}$$

We first interpret Equation (17), which decomposes the revenue rate into two parts associated with immediate consumption and future credit-based consumption, respectively. The immediate revenue (the first term) is earned either from a regular unit purchase (with revenue rate $\lambda \bar{w}_0 \bar{f} \bar{\pi}_0^s$) or from an immediate consumption after purchasing the pass (with revenue rate $\lambda \bar{w}_0 \bar{f} \bar{\pi}_0^p$). The revenue from credits (the second term) is the expected revenue from all credits, where the contribution of credit k to the revenue is $\Delta \bar{\eta}_k^w \bar{\pi}_k^p$.

In (18), we decompose the revenue from each credit further into a base revenue, $\bar{f}(\bar{\pi}_0^s + \bar{\pi}_0^p)$, which coincides with the immediate revenue per customer, and a term representing the contribution of capacity scarcity, $\big[\bar{\pi}_k^p - (\bar{\pi}_0^s + \bar{\pi}_0^p)\big] \eta^{z\star}$, which we call a *capacity-scarcity surcharge* (this term will

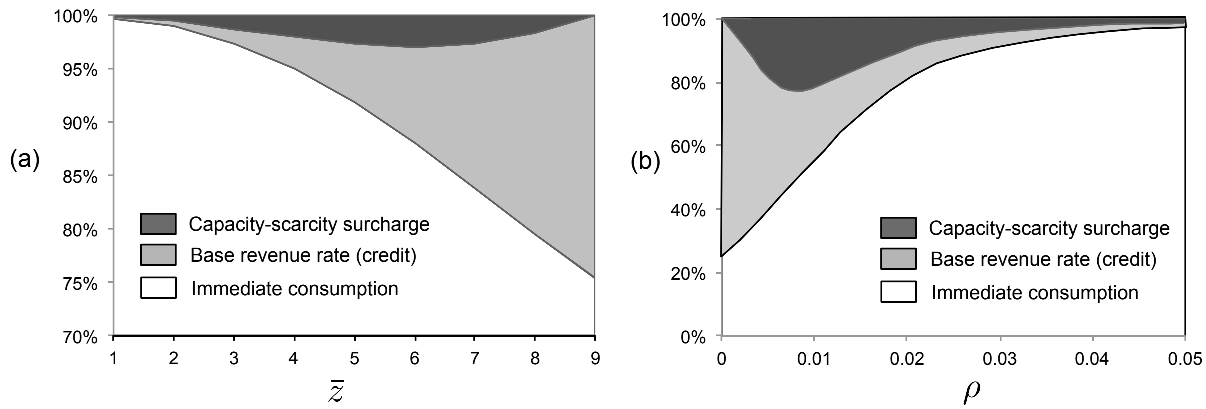

FIGURE 4. The percent composition of turnpike revenue rate as a function of (a) capacity $\bar{z}$ ($\rho = 0.05$), and (b) strategicity parameter ρ ($\bar{z} = 6$).

vanish if the capacity is ample, i.e., $\eta^{z\star} = 0$). Under limited capacity, Proposition 4 suggests that the difference of credit consumption and immediate consumption probabilities (i.e., $\bar{\pi}_k^p - (\bar{\pi}_0^s + \bar{\pi}_0^p)$) is positive if and only if the customers are not fully strategic. In that case, the revenue from a credit is strictly greater than the base immediate revenue. To summarize, the credit-based consumption generates a higher revenue rate than immediate consumption if the customers are not fully strategic *and* the capacity is limited.

In (19), the contribution of capacity-scarcity surcharge is aggregated over all credits, which represents the total extra revenue from credits. The total capacity-scarcity surcharge is proportional to the extra capacity consumed by the pass holders, namely, $\lambda \sum_{k=1}^{\bar{k}} \bar{w}_k \big[\bar{\pi}_k^p - (\bar{\pi}_0^s + \bar{\pi}_0^p)\big]$. If customers are fully strategic, then by Proposition 4, pass holders do not consume more capacity. In that case, the capacity scarcity makes no additional contribution to the revenue.

The structure of revenue is jointly modulated by the capacity constraint and strategic behavior. Figure 4 illustrates the fraction of the three terms in (19) with parameters $(\bar{k}, \tau, \lambda, a, a^P, \mu, \gamma, \beta) = (4, 120, 0.3, 1, 1.2, 2, 0.5, 1)$. The capacity scarcity may contribute up to 20% of the total revenue, but its contribution is not monotone in the capacity or strategic behavior. The contribution diminishes when the capacity is either too small or too large (as shown in Figure 4 (a)), and when the customers are too myopic or too strategic (as shown in Figure 4 (b)).

Figure 4 (a) shows that the credit's contribution to revenue is increasing in capacity. This occurs because, under increasingly limited capacity, the issuer tends to allocate more to customers who pay higher prices. Since the credit prices can be lower than the regular price, fewer passes should be sold and hence their contribution decreases.

The strategic behavior increases the credit's contribution to revenue, as shown in Figure 4 (b). This is because the pass sales are driven by the forward-looking behavior: myopic customers have little appreciation for future consumption, so they would prefer regular purchases to passes. Overall, Figure 4 suggests that passes contribute more revenue when the capacity is large and customers are more strategic.

Next, we compare the turnpike revenue rate before and after introducing passes. Let $\bar{R}^n(\infty)$ denote the turnpike revenue rate before the introduction of passes (benchmark), and recall that $\bar{R}(\infty)$ and $\bar{R}(0)$ are the turnpike revenue rate for myopic and fully strategic customers, respectively, after introducing passes. The following lemma suggests that passes can improve the revenue rate for both myopic customers and fully strategic customers.

Lemma 5. *When the capacity is ample ($\eta^{z\star} = 0$) and $\beta = 1$, we have $\bar{R}^n(\infty) = \bar{R}^n(0) < \min\{\bar{R}(\infty),\, \bar{R}(0)\}$.*

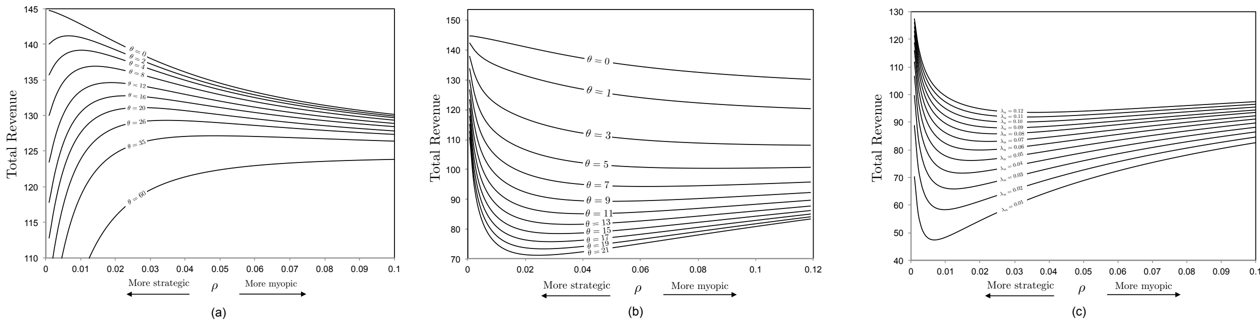

FIGURE 5. Comparison of revenues for different (ρ, β). (a) $\beta = e^{-0.001\theta}$ (b) $\beta = e^{-\rho\theta}$. (c) $\beta = \frac{\lambda_n}{\rho+\lambda_n}$.

7. Capitalizing on Strategic Behavior

Strategic behavior is known to hurt the revenue in many settings (see Aviv et al. (2009)). We now examine the circumstances in which passes can improve the revenue and how this potential improvement depends on strategic behavior. In particular, we consider joint influence of the level of strategic behavior ρ, post-consumption discount factor β, and total initial capacity $\bar{z}$. Both β and ρ describe discounting, but under different circumstances, and they may have different functional relationship. We consider three cases: (a) β is invariant with respect to ρ; (b) β is linked to ρ through the homogeneous subjective delay parameter θ as $\beta = e^{-\rho\theta}$; and (c) β is linked through the exponentially distributed subjective delay θ with intensity λ_n leading to $\beta = \mathbb{E}[e^{-\rho\theta}] = \frac{\lambda_n}{\lambda_n+\rho}$. For the invariant form (a) of the link, we used $\beta = e^{-0.001\theta}$ with the various fixed values of the subjective delay θ.

Figure 5 shows a general overview of possible behaviors of the total optimal revenues in terms of ρ, β and the forms of the link between them. The other parameters are fixed at the values $(\tau, \lambda, a, a^P, \mu, \gamma, \bar{z}, \bar{k}) = (500, 0.3, 1, 1, 2, 0.8, 50, 4)$ which also serve as a reference point for subsequent experiments of this section. An overall observation is that an increase in subjective delay leads to a lower revenue. On the other hand, the dependence of the revenue on ρ is generally not monotone, and there are different phenomena depending on the type of the link. For the case (a) of invariant β, very strategic behavior always hurts revenue unless θ is very close to zero. However, in a relatively wide range of θ, there is an increase in revenue with a decrease in strategic behavior for not-too-small ρ which can be interpreted as a capitalization on strategic behavior. Cases (b) and (c) are similar, with very strategic behavior always leading to capitalization. A smaller subjective delay also leads to wider ranges of capitalization. The difference of case (c) from (b) is a larger degree of uncertainty represented by the random subjective delay. We see that, in this case, the additional uncertainty significantly reduced the range of capitalization.

In the strategic frequent customer market, the observed capitalization effect is not always due to passes. A customer that has a perfect foresight, is infinitely patient, and does not worry about post-consumption uncertainty (i.e., $\rho = 0$ and $\beta = 1$) has the same expected future utility in each state if the system is in a steady state. This is true regardless of the presence of passes and makes such a consumer essentially myopic in behavior. In the absence of passes, the firm gets the same revenue both in the myopic case and in the fully strategic case without subjective discounting (this is supported by the result of Lemma 5). As ρ approaches zero and β approaches one, it is natural to expect some increase in revenue. The same effect is propagated to the case of passes. However, for the case of invariant β, increased strategic behavior always hurts the steady-state revenue in the absence of passes:

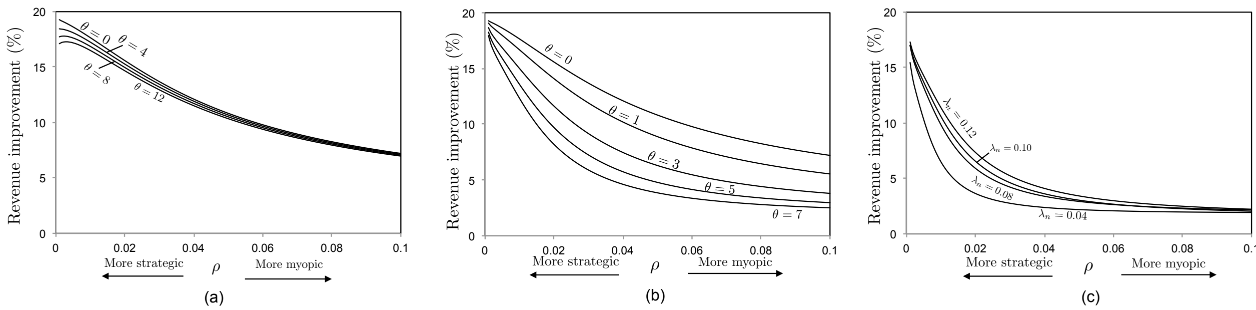

FIGURE 6. Percentage revenue improvement from passes (a) $\beta = e^{-0.001\theta}$ (b) $\beta = e^{-\rho\theta}$ (c) $\beta = \frac{\lambda_n}{\rho+\lambda_n}$.

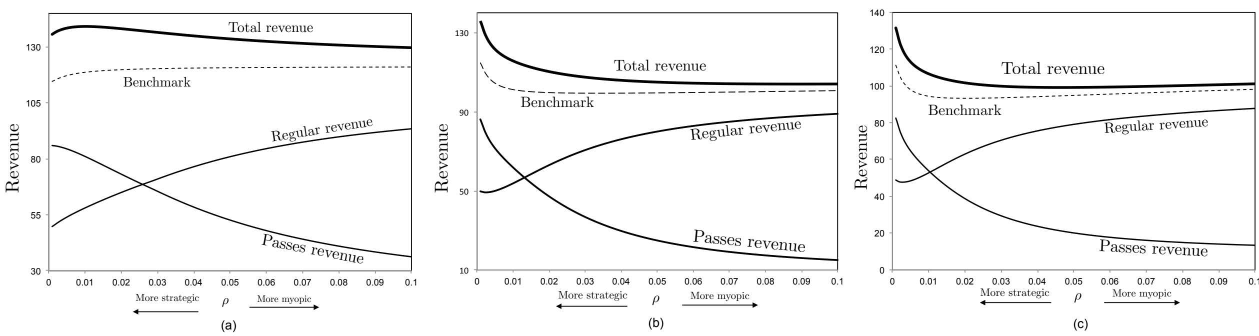

FIGURE 7. Comparison of revenues before and after introducing passes (a) $\beta = e^{-0.004}$ (b) $\beta = e^{-4\rho}$ (c) $\beta = \frac{0.16}{0.16+\rho}$.

Lemma 6. *When the capacity is ample ($\eta^z = 0$) and β is invariant with respect to ρ, the turnpike revenue rate in the absence of passes $\bar{R}^n(\rho)$ is increasing in ρ.*

Figure 5 (a) demonstrates that introduction of passes can reverse this trend as long as customers are not too strategic (i.e., when ρ is not too small) and subjective delay θ is not too large.

Next, we examine the percentage improvement in the total revenue compared with the benchmark. Figure 6 shows this percentage improvement as a function of ρ for three different types of the link between β and ρ and various values of the subjective delay. For all examined settings, the improvement increases as customers become more strategic (except for very low ρ in the invariant form of β, see Figure 6 (a)). Also, an increase in the subjective delay leads to a lower improvement. For this figure, we use $\gamma = 0.8$. A separate examination of smaller values of γ reveals the same trend except that the magnitudes of improvements are smaller (e.g., the maximum percentage improvement is 8% for the case of $\gamma = 0.3$).

Strategic behavior improves the revenue because it increases the pass sales as shown in Figure 7. In all three cases, we observe the shift of revenue from regular sales to passes as customers become more strategic. Moreover, the increase in pass revenues offsets losses in regular revenues resulting in the net total gain. The benchmark revenue on Figure 7 (a) is increasing in ρ, while it appears as decreasing in cases (b) and (c) within the range of ρ presented on the figure. In the latter two cases in is actually U-shaped as supported by the result of Lemma 5. While the lemma only directly covers $\beta = 1$, the stated equality between the extreme benchmark cases extends to any β if $\rho = 0$.

Figure 8 (a) examines how the capacity affects the percentage improvement in revenues from introduction of passes in the invariant discount case. As capacity becomes scarce, it becomes progressively more difficult to gain from passes and it takes more strategic behavior to reach a

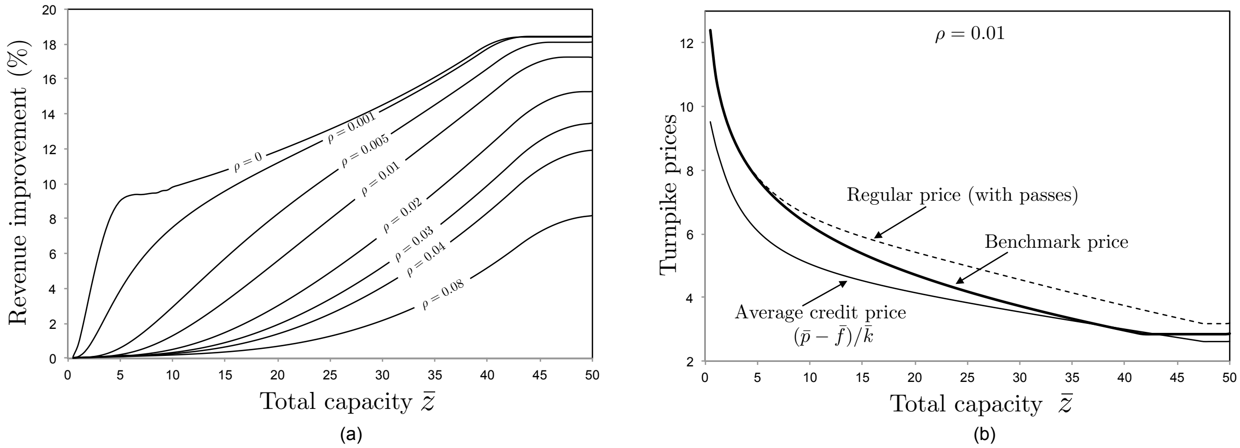

FIGURE 8. The percentage revenue improvement as a function of the total capacity ($\beta = e^{-0.004}$).

given level of percentage improvement in revenue. This improvement arises from a second-degree price discrimination mechanism by which the regular price is higher than the benchmark price, and the average credit price is typically lower than the benchmark price. When the capacity is scarce, the regular price increase is very small as shown in Figure 8 (b).

Capitalization phenomenon arises because the pass sales are driven by the forward-looking behavior of the customers. Passes require the customers to prepay for their future consumption, but such advance payment is not appealing to myopic customers who are not forward-looking. The quantity discount offered by passes is more likely to be appreciated by strategic customers who carefully plan future consumption. If the quantity discount offered by passes is sufficiently attractive, the customers need not wait to buy. As a result, the pass can encourage advance purchase when customers are strategic thereby mitigating strategic waiting. It has been discussed in the literature that the firm can utilize rationing strategies to create a sense of urgency for purchase (see Su (2007) and Liu and van Ryzin (2008)). Unlike the negative incentive of shortage risk, passes provide a new mechanism to capitalize on strategic behavior by offering a quantity discount for advance purchase and encouraging faster consumption.

Following this logic, one can speculate that, as customers become more strategic, the issuer should offer more credits on the pass to capture more revenue from forward-looking behavior. A supporting example is provided in Figure 9, in which the revenue rate is plotted as a function of the number of credits on the pass, $\bar{k}$, for different values of ρ. The figure indicates that, for customers who are more strategic (with smaller ρ), the optimal number of credits on a pass increases. Another observation in Figure 9 is that the strategic behavior improves the revenue across a wide range of $\bar{k}$ instead of only limited instances.

8. CONCLUSION

In summary, the credits on the pass can generate a higher revenue rate by selling more products at lower prices than regular sales, and this occurs only when the capacity is limited and the customers are not fully strategic. In total, the pass holders end up paying for their additional usage of capacity. Also, passes offer quantity discounts contingent on advance purchase, which mitigate strategic waiting and allow the issuer to benefit from strategic behavior.

The main modelling contribution of the paper is the joint dynamic pricing of passes and individual items under limited capacity and strategic customer behavior. The model also endogenizes the

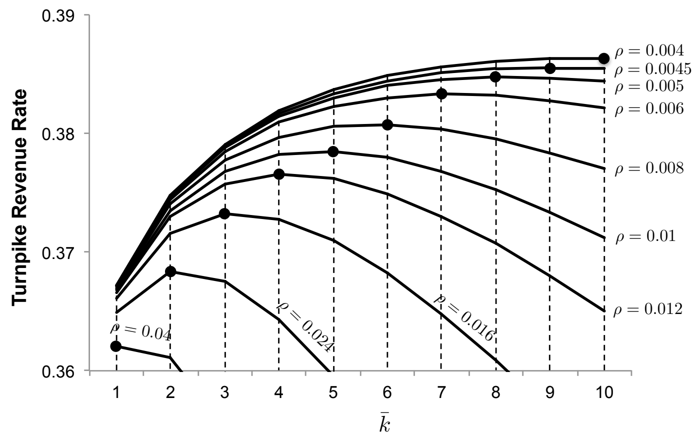

FIGURE 9. The turnpike revenue rate as a function of the number of credits $\bar{k}$ ($\beta = e^{-0.004}$).

customer credit utilization decisions. Modeling ideas and findings of this paper can motivate and inform further studies in the following areas:

- examination of short-term unlimited passes (e.g., the unlimited monthly pass offered by JetBlue) as an alternative to longer-term limited passes and general treatment of pass term and number of credits as decision variables;
- passes as a competitive tool in the presence of strategic customers;
- large-scale network pass pricing models and computational approaches; and
- empirical testing of customer behavior model and validation of the strategic customer assumption.

REFERENCES

K. L. Ailawadi and S. A. Neslin. The effect of promotion on consumption: Buying more and consuming it faster. *J. Marketing Res.*, 35(3):390–398, 1998.

S. Anderson, A. de Palma, and J. Thisse. *Discrete Choice Theory of Product Differentiation*. MIT Press, Cambridge, MA, 1992.

M. Andrews, G. Bruns, M. K. Dogru, and H. Lee. Understanding quota dynamics in wireless networks. Working paper, 2013.

J. L. Assuncao and R. J. Meyer. The rational effect of price promotions on sales and consumption. *Management Sci.*, 39(5):517–535, 1993.

Y. Aviv, Y. Levin, and M. Nediak. Counteracting strategic consumer behavior in dynamic pricing systems. In S. Netessine and C. Tang, editors, *Consumer-Driven Demand and Operations Management Models*, pages 323–352. Springer, 2009.

A. Bazhanov, Y. Levin, and M. Nediak. Quantity competition in the presence of strategic consumers. Working paper, 2014.

D. R. Bell, G. Iyer, and V. Padmanabhan. Price competition under stockpiling and flexible consumption. *J. Marketing Res.*, 39(3):292–303, 2002.

K. Bensinger. The frequent fliers who flew too much. Los Angeles Times, May 2012.

G. Cachon and R. Swinney. Purchasing, pricing, and quick response in the presence of strategic consumers. *Management Science*, 55(3):497–511, 2009.

J. C. Carbajo. The economics of travel passes: Non-uniform pricing in transport. *Journal of Transport Economics and Policy*, 22(2):153–173, 1988.

M. Cho, M. Fan, and Y. P. Zhou. Strategic consumer response to dynamic pricing of perishable products. In C.S. Tang and S. Netessine, editors, *Consumer-Driven Demand and Operations Management Models*. Springer, New York, NY., 2009.

C. S. Chu, P. Leslie, and A. Sorensen. Bundle-size pricing as an approximation to mixed bundling. *Amer. Econ. Rev.*, 101(1):263–303, 2011.

R. J. Dolan. Quantity discounts: managerial issues and research opportunities. *Marketing Sci.*, 6 (1):1–23, 1987.

D. Ellsberg. Risk, ambiguity, and the savage axioms. *Quarterly Journal of Economics*, 75(4): 643–669, 1961.

V. S Folkes, I. M. Martin, and K. Gupta. When to say when: Effects of supply on usage. *J. of Consumer Res.*, 20(3):467–477, 1993.

G. Gallego and G. van Ryzin. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Science*, 40(8):999–1020, 1994.

G. Gallego and R. Wang. Multiproduct price optimization and competition under the nested logit model with product-differentiated price sensitivities. *Oper. Res.*, 62(2):450–461, 2014.

Y. Levin, J. McGill, and M. Nediak. Dynamic pricing in the presence of strategic consumers and oligopolistic competition. *Management Science*, 55(1):32–46, 2009.

Y. Levin, M. Nediak, and A. Bazhanov. Quantity premiums and discounts in dynamic pricing. *Oper. Res.*, 62(4):846–863, 2014.

H. Li and W. T. Huh. Pricing multiple products with the multinomial logit and nested logit models: Concavity and implications. *Manuf. Serv. Oper. Man.*, 13(4):549–563, 2011.

J. Li, N. Granados, and S. Netessine. Are consumers strategic? structural estimation from the air-travel industry. *Management Sci.*, 60(9):2114–2137, 2014.

Q. Liu and G. van Ryzin. Strategic capacity rationing to induce early perchases. *Management Science*, 54(6):1115–1131, 2008.

S. Macé and S. A. Neslin. The determinants of pre- and postpromotion dips in sales of frequently purchased goods. *J. Marketing Res.*, 41(3):339–350, 2004.

E. J. Maskin and J. Riley. Monopoly with incomplete information. *RAND Journal of Economics*, 15(2):171–196, 1984.

D. McFadden. Modelling the choice of residential location. In A. Karlvist et al (eds.), editor, *Spatial Interaction Theory and Planning Models*, pages 75–96. North-Holland, Amsterdam, 1978.

W. Y. Oi. A disneyland dilemma: two-part tariffs for a mickey mouse monopoly. *Quart. J. Econom.*, 85(1):77–96, 1971.

C. Schlereth and B. Skiera. Measurement of consumer preferences for bucket pricing plans with different service attributes. *Intern. J. of Research in Marketing*, 29:167–180, 2012.

S. P. Sethi and G. L. Thompson. *Optimal Control Theory: Applications to Management Science and Economics*. Springer, 2000.

S. M. Shugan and J. Xie. Advance selling of service and other implication of separating purchase and consumption. *J. Service Res.*, 2:227–239, 2000.

L. A. Stole. Price discrimination and competition. In M. Armstrong and R. Porter, editors, *Handbook of Industrial Organization*, volume 3, chapter 34, pages 2221–2299. North Holland, 2007.

X. Su. Inter-temporal pricing with strategic customer behavior. *Management Science*, 53(5):726–741, 2007.

X. Su. Intertemporal pricing and consumer stockpiling. *Oper. Res.*, 58(4):1133–1147, 2010.

B. H. Sun, Y. Sun, and S. Li. When advance purchase need to be made for future consumption - an empirical investigation of consumer choice under bucket pricing. Working paper, 2006.

J. Tirole. *The Theory of Industrial Organization*. MIT Press, 1988.

R. Venkatesh and Vijay Mahajan. A probabilistic approach to pricing a bundle of products or services. *J. Marketing Res.*, 30(4):494–508, 1993.

B. Wansink. Can package size accelerate usage volume? *J. Marketing Res.*, 60(3):1–14, 1996.

H. C. W. L. Williams. On the formation of travel demand models and economic evaluation measures of user benefit. *Environment and Planning*, 9(3):285–344, 1977.

R. Wilson. *Nonlinear Pricing*. Oxford University Press, New York, 1993.

J. Xie and S. M. Shugan. Electronic tickets, smart cards, and online prepayments: when and how to advance sell. *Marketing Sci.*, 20(3):219–243, 2001.

Appendix

A. **Choice probability and utility equations for the passes model.**

A.1. *Choice probability.* In this section, we reproduce the fundamental assumptions and derivation of the nested logit model since these equations are needed for the analysis of the utility dynamics in the next section.

We define the function $G(x_1, x_2, x_3) = x_1 + (x_2^{\frac{1}{\gamma}} + x_3^{\frac{1}{\gamma}})^{\gamma}$ with derivatives

$$
\begin{aligned}
G_1(x_1, x_2, x_3) &= \frac{dG(x_1, x_2, x_3)}{dx_1} = 1, \\
G_2(x_1, x_2, x_3) &= \frac{dG(x_1, x_2, x_3)}{dx_2} = (x_2^{\frac{1}{\gamma}} + x_3^{\frac{1}{\gamma}})^{\gamma-1} x_2^{\frac{1}{\gamma}-1}, \\
G_3(x_1, x_2, x_3) &= \frac{dG(x_1, x_2, x_3)}{dx_3} = (x_2^{\frac{1}{\gamma}} + x_3^{\frac{1}{\gamma}})^{\gamma-1} x_3^{\frac{1}{\gamma}-1}.
\end{aligned}
$$

It is easy to verify that $G(x_1, x_2, x_3)$ is a homogeneous function with degree one and consequently its derivatives $G_i(x_1, x_2, x_3), i = 1, 2, 3$ are with degree zero. Let $u_{0t}^n = U_{0t} + \mu\epsilon_t^0, u_{0t}^s = a - f_t + \beta U_{0t} + \mu\epsilon_t^s, u_{0t}^p = a - p_t + \beta U_{\bar{k}t} + \mu\epsilon_t^p$ and assuming that

$$
\begin{aligned}
&\Pr(\epsilon_t^0 < y_1, \epsilon_t^s < y_2, \epsilon_t^p < y_3) \\
=&F(y_1, y_2, y_3) = \exp[-G(e^{-y_1}, e^{-y_2}, e^{-y_3})]
\end{aligned}
$$

Its derivatives are

$$
F_i'(y_1, y_2, y_3) = \frac{dF(y_1, y_2, y_3)}{dy_i} = F(y_1, y_2, y_3) G_i(e^{-y_1}, e^{-y_2}, e^{-y_3}) e^{-y_i}, i = 1, 2, 3.
$$

Then we derive the regular ticket choice probability as the following

$$
\begin{aligned}
\pi_{0t}^s =& \Pr(u_{0t}^n < u_{0t}^s, u_{0t}^p < u_{0t}^s) \\
=& \int_{-\infty}^{\infty} F_2'\Big(\frac{a - f_t - (1-\beta)U_{0t}}{\mu} + \epsilon, \epsilon, \frac{p_t - f_t - \beta U_{\bar{k}t} + \beta U_{0t}}{\mu} + \epsilon\Big) d\epsilon \\
=& \int_{-\infty}^{\infty} \exp[-G(e^{-\frac{a-f_t-(1-\beta)U_{0t}}{\mu}-\epsilon}, e^{-\epsilon}, e^{-\frac{p_t-f_t-\beta U_{\bar{k}t}+\beta U_{0t}}{\mu}-\epsilon})] \\
& G_2(e^{-\frac{a-f_t-(1-\beta)U_{0t}}{\mu}-\epsilon}, e^{-\epsilon}, e^{-\frac{p_t-f_t-\beta U_{\bar{k}t}+\beta U_{0t}}{\mu}-\epsilon}) e^{-\epsilon} d\epsilon, \\
=& \int_{-\infty}^{\infty} \exp[-e^{-\epsilon} e^{-\frac{a-f_t-(1-\beta)U_{0t}}{\mu}} G(1, e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}, e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}})] \\
& G_2(1, e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}, e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}}) e^{-\epsilon} d\epsilon \\
=& \frac{e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}} G_2(1, e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}, e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}})}{G(1, e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}, e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}})} \\
=& \frac{\exp(\frac{a-f_t-(1-\beta)U_{0t}}{\mu\gamma}) \Big[\exp(\frac{a-f_t-(1-\beta)U_{0t}}{\mu\gamma}) + \exp(\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu\gamma})\Big]^{\gamma-1}}{1 + \Big[\exp(\frac{a-f_t-(1-\beta)U_{0t}}{\mu\gamma}) + \exp(\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu\gamma})\Big]^{\gamma}}
\end{aligned}
\tag{20}
$$

where the third equality follows from the homogeneity of degree one of G, and the homogeneity of degree zero of G_2. This argument can be also applied to prove the formula for π_{0t}^p.

A.2. *Proof of Lemma 1.* First, consider the utility U_{0t} over a short interval $[t, t+\delta]$, it satisfies the following equation

$$\begin{aligned} U_{0t} &= (1-e^{-\lambda\delta})\mathbb{E}\big[\max\{u^n_{0(t+\delta)}, u^s_{0(t+\delta)}, u^p_{0(t+\delta)}\}\big] + e^{-\lambda\delta}e^{-\rho\delta}\mathbb{E}[u^n_{0(t+\delta)}] \\ &= (1-e^{-\lambda\delta})\mathbb{E}\big[\max\{u^n_{0(t+\delta)}, u^s_{0(t+\delta)}, u^p_{0(t+\delta)}\} - e^{-\rho\delta}u^n_{0(t+\delta)}\big] + e^{-\rho\delta}\mathbb{E}[u^n_{0(t+\delta)}]. \end{aligned}$$

Then we obtain the differential equation

$$\begin{aligned} \dot{U}_{0t} &= \lim_{\delta\to 0}\frac{U_{0(t+\delta)} - U_{0t}}{\delta} \\ &= \lim_{\delta\to 0}\frac{1}{\delta}\Big(U_{0(t+\delta)} - (1-e^{-\lambda\delta})\mathbb{E}\big[\max\{u^n_{0(t+\delta)}, u^s_{0(t+\delta)}, u^p_{0(t+\delta)}\} - e^{-\rho\delta}u^n_{0(t+\delta)}\big] - e^{-\rho\delta}\mathbb{E}[u^n_{0(t+\delta)}]\Big) \\ &= -\lim_{\delta\to 0}\frac{1-e^{-\lambda\delta}}{\delta}\mathbb{E}\big[\max\{u^n_{0t}, u^s_{0t}, u^p_{0t}\} - u^n_{0t}\big] + \lim_{\delta\to 0}\frac{U_{0(t+\delta)} - e^{-\rho\delta}\mathbb{E}[u^n_{0(t+\delta)}]}{\delta} \\ &= -\lambda\mathbb{E}\big[\max\{u^n_{0t}, u^s_{0t}, u^p_{0t}\} - u^n_{0t}\big] + \lim_{\delta\to 0}\frac{1-e^{-\rho\delta}}{\delta}U_{0(t+\delta)} \\ &= -\lambda\mathbb{E}\big[\max\{u^n_{0t}, u^s_{0t}, u^p_{0t}\} - u^n_{0t}\big] + \rho U_{0t}. \end{aligned} \tag{21}$$

The above equation can be interpreted in terms of the forward-looking behaviour. It suggests that the rate of change in utility is modulated by the overall quote request intensity $\lambda$and, for each arrival, results in the average change in utility given by the expectation operator. Expressions inside the expectation operator are differences in utility of the best possible option relative to the no-purchase option.

The next step is to find an explicit expression for $\mathbb{E}\big[\max\{u^n_{0t}, u^s_{0t}, u^p_{0t}\} - u^n_{0t}\big]$ by examining the expected maximum utility

$$\begin{aligned} \mathbb{E}[\bar{U}] =& \mathbb{E}[\max\{u^n_{0t}, u^s_{0t}, u^p_{0t}\}] \\ =& \int_{-\infty}^{\infty}(U_{0t}+\mu\epsilon)F_1'\Big(\epsilon, -\frac{a-f_t-(1-\beta)U_{0t}}{\mu}+\epsilon, -\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}+\epsilon\Big)d\epsilon \\ &+ \int_{-\infty}^{\infty}(a-f_t+\beta U_{0t}+\mu\epsilon)F_2'\Big(\frac{a-f_t-(1-\beta)U_{0t}}{\mu}+\epsilon, \epsilon, \frac{p_t-f_t-\beta U_{\bar{k}t}+\beta U_{0t}}{\mu}+\epsilon\Big)d\epsilon \\ &+ \int_{-\infty}^{\infty}(a-p_t+\beta U_{\bar{k}t}+\mu\epsilon)F_3'\Big(\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}+\epsilon, \frac{f_t-p_t+\beta U_{\bar{k}t}-\beta U_{0t}}{\mu}+\epsilon, \epsilon\Big)d\epsilon \\ =& U_{0t} + \int_{-\infty}^{\infty}(\mu\epsilon)F_1'\Big(\epsilon, -\frac{a-f_t-(1-\beta)U_{0t}}{\mu}+\epsilon, -\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}+\epsilon\Big)d\epsilon \\ &+ \int_{-\infty}^{\infty}(a-f_t-(1-\beta)U_{0t}+\mu\epsilon)F_2'\Big(\frac{a-f_t-(1-\beta)U_{0t}}{\mu}+\epsilon, \epsilon, \frac{p_t-f_t-\beta U_{\bar{k}t}+\beta U_{0t}}{\mu}+\epsilon\Big)d\epsilon \\ &+ \int_{-\infty}^{\infty}(a-p_t+\beta U_{\bar{k}t}-U_{0t}+\mu\epsilon)F_3'\Big(\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}+\epsilon, \frac{f_t-p_t+\beta U_{\bar{k}t}-\beta U_{0t}}{\mu}+\epsilon, \epsilon\Big)d\epsilon \end{aligned}$$

where

$$F_1'\Big(\epsilon, -\frac{a-f_t-(1-\beta)U_{0t}}{\mu}+\epsilon, -\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}+\epsilon\Big)$$
$$= \exp\Big[-e^{-\epsilon}G(1, e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}, e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}})\Big]G_1(1, e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}, e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}})e^{-\epsilon}$$
$$F_2'\Big(\frac{a-f_t-(1-\beta)U_{0t}}{\mu}+\epsilon, \epsilon, \frac{p_t-f_t-\beta U_{\bar{k}t}+\beta U_{0t}}{\mu}+\epsilon\Big)$$
$$= \exp\Big[-e^{-\frac{a-f_t-(1-\beta)U_{0t}}{\mu}-\epsilon}G(1, e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}, e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}})\Big]G_2(1, e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}, e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}})e^{-\epsilon}$$
$$F_3'\Big(\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}+\epsilon, \frac{f_t-p_t+\beta U_{\bar{k}t}-\beta U_{0t}}{\mu}+\epsilon, \epsilon\Big)$$
$$= \exp\Big[-e^{-\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}-\epsilon}G(1, e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}, e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}})\Big]G_3(1, e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}, e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}})e^{-\epsilon} \tag{22}$$

Let $w_1 = \mu\epsilon, w_2 = a - f_t - (1-\beta)U_{0t} + \mu\epsilon, w_3 = a - p_t + \beta U_{\bar{k}t} - U_{0t} + \mu\epsilon$, then (22) becomes

$$\begin{aligned}\mathbb{E}[\bar{U}] =& U_{0t} + \int_{-\infty}^{\infty} w_1 \exp\Big[-e^{-\frac{w_1}{\mu}}G\Big]G_1(1, e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}, e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}})e^{-\frac{w_1}{\mu}}d\frac{w_1}{\mu}\\
&+\int_{-\infty}^{\infty} w_2 \exp\Big[-e^{-\frac{w_2}{\mu}}G\Big]e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}G_2(1, e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}, e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}})e^{-\frac{w_2}{\mu}}d\frac{w_2}{\mu}\\
&+\int_{-\infty}^{\infty} w_3 \exp\Big[-e^{-\frac{w_3}{\mu}}G\Big]e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}}G_3(1, e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}, e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}})e^{-\frac{w_3}{\mu}}d\frac{w_3}{\mu}\end{aligned}$$

It can be verified that

$$\begin{aligned}&G(1, e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}, e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}})\\
=&G_1(1, e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}, e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}}) + e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}G_2(1, e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}, e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}})\\
&+e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}}G_3(1, e^{\frac{a-f_t-(1-\beta)U_{0t}}{\mu}}, e^{\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu}})\end{aligned}$$

Then, by symmetry, we have

$$\begin{aligned}\mathbb{E}[\bar{U}] =& U_{0t} + \int_{-\infty}^{\infty} w \exp\big(-e^{-\frac{w}{\mu}}G\big)Ge^{-\frac{w}{\mu}}d\frac{w}{\mu} \overset{y=Ge^{-\frac{w}{\mu}}}{=} U_{0t} - \int_0^{+\infty}\mu(\ln y - \ln G)e^{-y}dy\\
=& U_{0t} + \mu\ln G - \mu\int_0^{+\infty}e^{-y}\ln y dy = U_{0t} + \mu\ln G + \mu\gamma_{em}\\
=& U_{0t} + \mu\ln\Big\{1 + \Big[\exp\Big(\frac{a-f_t-(1-\beta)U_{0t}}{\mu\gamma}\Big) + \exp\Big(\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu\gamma}\Big)\Big]^{\gamma}\Big\} + \mu\gamma_{em}\end{aligned}$$

in which γ_{em} is the Euler-Mascheroni constant (0.57726...)

Since $\mathbb{E}[\bar{U} - u_t^0] = \mathbb{E}[\bar{U}] - (U_{0t} + \mu\gamma_{em})$ (because the marginal distribution of ϵ_t^0 is a univariate extreme value distribution with mean $\mu\gamma_{em}$), we have $\mathbb{E}\big[\max\{u_{0t}^n, u_{0t}^s, u_{0t}^p\} - u_{0t}^n\big] = \mu\ln\Big\{1 + \Big[\exp\Big(\frac{a-f_t-(1-\beta)U_{0t}}{\mu\gamma}\Big) + \exp\Big(\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu\gamma}\Big)\Big]^{\gamma}\Big\}$. Substituting this expression into (21) gives

$$\begin{aligned}\dot{U}_{0t} &= -\lambda\mu\ln\Big\{1 + \Big[\exp\Big(\frac{a-f_t-(1-\beta)U_{0t}}{\mu\gamma}\Big) + \exp\Big(\frac{a-p_t+\beta U_{\bar{k}t}-U_{0t}}{\mu\gamma}\Big)\Big]^{\gamma}\Big\} + \rho U_{0t}\\
&= \lambda\mu\ln(1 - \pi_{0t}^p - \pi_{0t}^s) + \rho U_{0t},\end{aligned} \tag{23}$$

where the second equality follows from (1) and (2). Thus, we have proved (7).

One can repeat the above procedure for $U_{kt}, k = 1, \ldots, \bar{k}$, to obtain the following differential equation

$$\dot{U}_{kt} \;=\; -\lambda \mathbb{E}\Big[\max\Big\{u^n_{kt}, u^p_{kt}\Big\} - u^n_{kt}\Big] + \rho U_{kt} = \lambda\mu \ln(1-\pi^p_{kt}) + \rho U_{kt}. \tag{24}$$

Then, the equation (8) and (9) in this lemma directly follows from (23) and (24).

It remains to establish the signs of utility. From 23, it is immediate to see that, after substitution $U_{0t} = e^{\rho t} V_{0t}$, we formally get

$$\dot{V}_{0t} = e^{-\rho t}\lambda\mu \ln(1-\pi^p_{0t}-\pi^s_{0t}) \leq 0$$

for all t while $V_{0\tau} = 0$, and, therefore, V_{0t} is non-negative along with U_{0t}. For $\Delta U_{kt} \geq 0,\ k = 1, \ldots, \bar{k}$, we employ a relaxation argument. Indeed, U_{kt} is precisely the expected utility of all purchases occuring from t to τ given that the customer currently holds k credits. Since consumption with each additional credit has, on average, a positive expected contribution to utility, the utility decreases in the number of credits: $U_{kt} \geq U_{(k-1)t}$. Therefore, $\Delta U_{kt} = U_{kt} - \beta U_{(k-1)t} \geq U_{kt} - U_{(k-1)t} \geq 0$.

B. **Proof of Lemma 2.** We first show that w_{0t} has a positive lower bound, i.e., $w_{0t} > \exp(-\lambda\tau) > 0$. To begin with, consider the differential equation $\dot{w}_t = -\lambda w_t$ with boundary condition $w_0 = 1$, the solution is an exponential curve $w_t = \exp(-\lambda t)$, which has a lower bound in the finite horizon $[0,\tau]$, namely $w_t \geqslant \exp(-\lambda\tau)$ for $t \in [0,\tau]$. Next, consider the state equation for w_{0t}:

$$\dot{w}_{0t} = \lambda w_{1t}\pi^p_{1t} - \lambda w_{0t}\pi^p_{0t} \geqslant -\lambda w_{0t}\pi^p_{0t} \geqslant -\lambda w_{0t},$$

Therefore, we obtain $w_{0t} \geqslant w_t > \exp(-\lambda\tau) > 0$.

Consider the nonlinear terms of the Hamiltonian

$$H_n = \lambda w_{0t}(p_t\pi^p_{0t} + f_t\pi^s_{0t}) - \lambda\mu(\beta\eta^u_{1t} - \eta^u_{0t})\ln(1-\pi^p_{0t}-\pi^s_{0t})$$

The Hessian of the Hamiltonian H, denoted as $\mathcal{H}$, can be calculated as the following

$$\mathcal{H}(1,1) = \frac{\partial^2 H_n}{\partial(\pi^p_{0t})^2} = -\lambda\mu w_{0t}\frac{\pi^p_{0t}\pi^s_{0t} - \pi^p_{0t}\frac{\beta\eta^u_{1t}-\eta^u_{0t}}{w_{0t}} + (1-\pi^s_{0t})^2}{(1-\pi^p_{0t}-\pi^s_{0t})^2\pi^p_{0t}},$$

$$\mathcal{H}(2,2) = \frac{\partial^2 H_n}{\partial(\pi^s_{0t})^2} = -\lambda\mu w_{0t}\frac{\pi^p_{0t}\pi^s_{0t} - \pi^s_{0t}\frac{\beta\eta^u_{1t}-\eta^u_{0t}}{w_{0t}} + (1-\pi^p_{0t})^2}{(1-\pi^p_{0t}-\pi^s_{0t})^2\pi^s_{0t}},$$

$$\mathcal{H}(1,2) = \mathcal{H}(2,1) = \frac{\partial^2 H_n}{\partial\pi^p_{0t}\partial\pi^s_{0t}} = -\lambda\mu w_{0t}\frac{(2-\pi^p_{0t}-\pi^s_{0t}-\frac{\beta\eta^u_{1t}-\eta^u_{0t}}{w_{0t}})}{(1-\pi^p_{0t}-\pi^s_{0t})^2},$$

When $\beta\eta^u_{1t} - \eta^u_{0t} < w_{0t}$, $\pi^p_{0t}\pi^s_{0t} - \pi^p_{0t}\frac{\beta\eta^u_{1t}-\eta^u_{0t}}{w_{0t}} + (1-\pi^s_{0t})^2 > \pi^p_{0t}\pi^s_{0t} - \pi^p_{0t} + (1-\pi^s_{0t})^2 = (1-\pi^s_{0t})(1-\pi^s_{0t}-\pi^p_{0t}) \geq 0$, and, thus, $\mathcal{H}(1,1) < 0$. Similarly, $\beta\eta^u_{1t} - \eta^u_{0t} < w_{0t}$ implies that $\mathcal{H}(2,2) < 0$.

Since $\mathcal{H}(1,1) < 0$, and the determinant of $\mathcal{H}$ is

$$\begin{aligned}\det(\mathcal{H}) &= \mathcal{H}(1,1)\times\mathcal{H}(2,2) - \mathcal{H}(1,2)\times\mathcal{H}(2,1)\\ &= \lambda^2\mu^2 w^2_{0t}\frac{[1-(\pi^p_{0t}+\pi^s_{0t})\frac{\beta\eta^u_{1t}-\eta^u_{0t}}{w_{0t}}]}{\pi^p_{0t}\pi^s_{0t}(1-\pi^p_{0t}-\pi^s_{0t})^2} > 0,\end{aligned}$$

$\mathcal{H}$ is negative definite and the Hamiltonian is concave.

C. **The Turnpike (steady-state) Equations.** After the change of variables, the adjoint equations for the population distribution become

$$\Delta\dot{\eta}^w_{1t} = -\lambda\big[(\pi^p_{1t} - \pi^s_{0t} - \pi^p_{0t})\eta^z - \pi^p_{1t}\Delta\eta^w_{1t} - \pi^p_{0t}\Delta\eta^w_{0t} + f_t\pi^s_{0t} + p_t\pi^p_{0t}\big], \tag{25}$$

$$\Delta\dot{\eta}^w_{kt} = -\lambda\big[(\pi^p_{kt} - \pi^p_{(k-1)t})\eta^z - \pi^p_{kt}\Delta\eta^w_{kt} + \pi^p_{(k-1)t}\Delta\eta^w_{(k-1)t}\big],\ k = 2, \ldots, \bar{k}. \tag{26}$$

with boundary conditions $\Delta\eta^w_{k\tau} = 0, k = 1, \ldots, \bar{k}$.

Proposition 7. *The turnpike is characterized by the following equations:*

$$\bar{w}_{(k+1)}\bar{\pi}^p_{(k+1)} = \bar{w}_k\bar{\pi}^p_k, \qquad k = 0,\dots,\bar{k}-1, \tag{27}$$

$$\lambda\mu\ln(1-\bar{\pi}^p_0-\bar{\pi}^s_0) = -\rho\bar{U}_0, \tag{28}$$

$$\lambda\mu\big[\ln(1-\bar{\pi}^p_1)-\ln(1-\bar{\pi}^p_0-\bar{\pi}^s_0)\big] = -\rho\Delta\bar{U}_1, \tag{29}$$

$$\lambda\mu\big[\ln(1-\bar{\pi}^p_k)-\ln(1-\bar{\pi}^p_{(k-1)})\big] = -\rho\Delta\bar{U}_k, \quad k = 2,\dots,\bar{k}. \tag{30}$$

$$\bar{f}\bar{\pi}^s_0+\bar{p}\bar{\pi}^p_0 = (\bar{\pi}^s_0+\bar{\pi}^p_0-\bar{\pi}^p_1)\eta^{z\star}+\bar{\pi}^p_0\Delta\bar{\eta}^w_0+\bar{\pi}^p_1\Delta\bar{\eta}^w_1, \tag{31}$$

$$\bar{\pi}^p_k(\Delta\bar{\eta}^w_k-\eta^{z\star}) = \bar{\pi}^p_{(k-1)}(\Delta\bar{\eta}^w_{(k-1)}-\eta^{z\star}), \quad k = 2,\dots,\bar{k}. \tag{32}$$

$$\lambda\bar{w}_0\big[(1-\beta^{\bar{k}+1})\bar{\pi}^p_0+(1-\beta)\bar{\pi}^s_0\big] = \rho\bar{\eta}^u_0 \tag{33}$$

$$\bar{w}_0\bar{\pi}^p_0\beta^{\bar{k}-k+1}+\tfrac{1}{\mu}\bar{w}_k\bar{\pi}^p_k(1-\bar{\pi}^p_k)(\eta^{z\star}-\Delta\bar{\eta}^w_k)-\bar{\pi}^p_k(\beta\bar{\eta}^u_{(k+1)}-\bar{\eta}^u_k)+\tfrac{\rho}{\lambda}\bar{\eta}^u_k = 0, \ k = 1,\dots,\bar{k}-1, \tag{34}$$

$$\bar{w}_0\bar{\pi}^p_0\beta+\tfrac{1}{\mu}\bar{w}_{\bar{k}}\bar{\pi}^p_{\bar{k}}(1-\bar{\pi}^p_{\bar{k}})(\eta^{z\star}-\Delta\bar{\eta}^w_{\bar{k}})+\bar{\eta}^u_{\bar{k}}\bar{\pi}^p_{\bar{k}}+\tfrac{\rho}{\lambda}\bar{\eta}^u_{\bar{k}} = 0, \tag{35}$$

$$\bar{w}_0\Big(\bar{f}-\eta^{z\star}-\tfrac{\mu}{1-\bar{\pi}^p_0-\bar{\pi}^s_0}\Big)+(\beta\bar{\eta}^u_1-\bar{\eta}^u_0)\tfrac{\mu}{1-\bar{\pi}^p_0-\bar{\pi}^s_0} = 0, \tag{36}$$

$$\bar{w}_0\Big(\bar{p}-\Delta\bar{\eta}^w_0-\eta^{z\star}-\tfrac{\mu}{1-\bar{\pi}^p_0-\bar{\pi}^s_0}\Big)+(\beta\bar{\eta}^u_1-\bar{\eta}^u_0)\tfrac{\mu}{1-\bar{\pi}^p_0-\bar{\pi}^s_0} = 0, \tag{37}$$

where $\bar{f},\bar{p}$ are given by

$$\bar{f} = a+\mu\gamma\Big[\ln(1-\bar{\pi}^p_0-\bar{\pi}^s_0)-\ln\bar{\pi}^s_0\Big]+\mu(1-\gamma)\Big[\ln(1-\bar{\pi}^p_0-\bar{\pi}^s_0)-\ln(\bar{\pi}^p_0+\bar{\pi}^s_0)\Big]-(1-\beta)\bar{U}_0, \tag{38}$$

$$\begin{aligned}\bar{p} = {} & a+\mu\gamma\Big[\ln(1-\bar{\pi}^p_0-\bar{\pi}^s_0)-\ln\bar{\pi}^p_0\Big]+\mu(1-\gamma)\Big[\ln(1-\bar{\pi}^p_0-\bar{\pi}^s_0)-\ln(\bar{\pi}^p_0+\bar{\pi}^s_0)\Big] \\ & +\sum_{k=1}^{\bar{k}}\beta^{\bar{k}-k+1}\Delta\bar{U}_k-(1-\beta^{\bar{k}+1})\bar{U}_0.\end{aligned} \tag{39}$$

The equations (31) and (32) are obtained by setting the derivatives in (25), (26) to zero.

D. **Proof of Proposition 4.** For $\rho = 0$, (29) and (30) imply that $\bar{\pi}^s_0+\bar{\pi}^p_0 = \bar{\pi}^p_1 = \cdots = \bar{\pi}^p_{\bar{k}}$. Then, by 27, we obtain $\bar{w}_0 = \bar{w}_1 = \bar{w}_2 = \cdots = \bar{w}_{\bar{k}}$. For $0<\rho<\infty$, since $\Delta\bar{U}_k>0, k = 1,\dots,\bar{k}$, then by (29) and (30) we have $\bar{\pi}^s_0+\bar{\pi}^p_0<\bar{\pi}^p_1<\cdots<\bar{\pi}^p_{\bar{k}}$. Thereby completing the proof.

E. **Proof of Corollary 1.** By Proposition 4, we have $\bar{\pi}^p_{(k-1)}\leqslant\bar{\pi}^p_k$. According to Proposition 5, we have $\bar{f}(\bar{\pi}^s_0+\bar{\pi}^p_0)-\bar{\pi}^p_k\Delta\bar{\eta}^w_k = \eta^{z\star}(\bar{\pi}^s_0+\bar{\pi}^p_0-\bar{\pi}^p_k)\leqslant 0$ for $k = 1,\dots,\bar{k}$, and $\bar{\pi}^p_{(k-1)}\Delta\bar{\eta}^w_{(k-1)} = \bar{\pi}^p_k\Delta\bar{\eta}^w_k+\eta^{z\star}(\bar{\pi}^p_{(k-1)}-\bar{\pi}^p_k)\leqslant\bar{\pi}^p_k\Delta\bar{\eta}^w_k$ for $k = 2,\dots,\bar{k}$, because $\eta^{z\star}\geqslant 0$. It is clear that the equality holds if $\eta^{z\star} = 0$ or $\bar{\pi}^s_0+\bar{\pi}^p_0 = \bar{\pi}^p_1 = \cdots = \bar{\pi}^p_{\bar{k}}$, the condition for the latter is $\rho = 0$, which follows again from Proposition 4.

F. **Proof of Proposition 6.** The turnpike revenue rate is

$$\begin{aligned}\bar{R} &= \lambda\bar{w}_0(\bar{f}\bar{\pi}^s_0+\bar{p}\bar{\pi}^p_0) \\ &= \lambda\bar{w}_0\bar{f}(\bar{\pi}^s_0+\bar{\pi}^p_0)+\lambda\bar{w}_0\Delta\bar{\eta}^w_0\bar{\pi}^p_0 \\ &= \lambda\bar{w}_0\bar{f}(\bar{\pi}^s_0+\bar{\pi}^p_0)+\lambda\bar{w}_0\bar{\pi}^p_0\sum_{k=1}^{\bar{k}}\Delta\bar{\eta}^w_k \\ &= \lambda\bar{w}_0\bar{f}(\bar{\pi}^s_0+\bar{\pi}^p_0)+\lambda\sum_{k=1}^{\bar{k}}\bar{w}_k\bar{\pi}^p_k\Delta\bar{\eta}^w_{kt},\end{aligned} \tag{40}$$

thereby proving the first equation in the proposition. By Proposition 5, we have

$$\bar{\pi}_k^p \Delta\bar{\eta}_k^w = \bar{f}(\bar{\pi}_0^s + \bar{\pi}_0^p) + \big(\bar{\pi}_k^p - (\bar{\pi}_0^s + \bar{\pi}_0^p)\big)\eta^{z\star}, \quad k = 1, \ldots, \bar{k}.$$

Combining the above equation with (40) leads to the second equation in this proposition.

G. **Proof of Proposition 2.** For myopic customers, the utility terms vanish, i.e., $U_{kt} = 0$, for all k and $t \in [0, \tau]$. The Hamiltonian reduces to

$$H = \lambda w_{0t}\big[(f_t - \eta^{z\star})\pi_{0t}^s + (p_t - \Delta\eta_{0t}^w - \eta^{z\star})\pi_{0t}^p\big] + \pi_1^p \sum_{k=1}^{\bar{k}} w_{kt}(\Delta\eta_{kt}^w - \eta^{z\star}).$$

The adjoint equations become

$$\begin{aligned}
\Delta\dot{\eta}_{1t}^w &= -\lambda\big[(\pi_1^p - \pi_{0t}^s - \pi_{0t}^p)\eta^{z\star} - \pi_1^p \Delta\eta_{1t}^w - \pi_{0t}^p \Delta\eta_{0t}^w + f_t \pi_{0t}^s + p_t \pi_{0t}^p\big],\\
\Delta\dot{\eta}_{kt}^w &= -\lambda\pi_1^p\big[-\Delta\eta_{kt}^w + \Delta\eta_{(k-1)t}^w\big],\ k = 2, \ldots, \bar{k}.
\end{aligned}$$

with boundary conditions $\Delta\eta_{k\tau}^w = 0, k = 1, \ldots, \bar{k}$.

Since the Hamiltonian is jointly concave in (π_{0t}^p, π_{0t}^s), we obtain the first order optimality condition as the following:

$$\Big\{\frac{\partial H}{\partial \pi_{0t}^p} = 0, \frac{\partial H}{\partial \pi_{0t}^s} = 0\Big\} \Rightarrow p_t = f_t + \Delta\eta_{0t}^w,\ f_t - \eta^{z\star} = \frac{\mu}{1 - \pi_{0t}^p - \pi_{0t}^s}. \tag{41}$$

Expand the terms π_{0t}^p, π_{0t}^s, then the optimal regular ticket price f_t satisfies the following equation

$$f_t - \eta^{z\star} = \mu\Big\{1 + \Big[\exp\Big(\frac{a - f_t}{\mu\gamma}\Big) + \exp\Big(\frac{a - f_t - \Delta\eta_{0t}^w}{\mu\gamma}\Big)\Big]^{\gamma}\Big\}. \tag{42}$$

(41) also enables us to rewrite the equation for the adjoint variable $\Delta\eta_{1t}^w$ as the following

$$\Delta\dot{\eta}_{1t}^w = -\lambda(f_t - \mu - \eta^{z\star}) + \lambda\pi_1^p(\Delta\eta_{1t}^w - \eta^{z\star}).$$

In the turnpike, $\Delta\dot{\eta}_{kt}^w = 0,\ k = 1, \ldots, \bar{k}$, which leads to $\Delta\bar{\eta}_1^w = \ldots = \Delta\bar{\eta}_{\bar{k}}^w$ and

$$\bar{f} - \mu - \pi_1^p \Delta\bar{\eta}_{kt}^w = \eta^{z\star}(1 - \pi_1^p),\ k = 1, \ldots, \bar{k}. \tag{43}$$

By substituting $\pi_{1t}^p = \frac{\exp(a^P/\mu)}{1+\exp(a^P/\mu)}$ into the above equation, we obtain the expression for $\Delta\bar{\eta}_{kt}^w$ as a function of $\bar{f}$, as shown in the proposition.

Substituting (43) into (42) gives the equation that fully characterizes $\bar{f}$:

$$\begin{aligned}
\bar{f} - \eta^{z\star} &= \mu\Big\{1 + \Big[\exp\Big(\frac{a - \bar{f}}{\mu\gamma}\Big) + \exp\Big(\frac{a - \bar{f} - \bar{k}\Delta\bar{\eta}_1^w}{\mu\gamma}\Big)\Big]^{\gamma}\Big\}\\
&= \mu\Big\{1 + \Big[\exp\Big(\frac{a - \bar{f}}{\mu\gamma}\Big) + \exp\Big(\frac{a - \bar{f} - \bar{k}(\bar{f} - \mu)(1 + e^{-a^P/\mu}) + \bar{k}\eta^{z\star}e^{-a^P/\mu}}{\mu\gamma}\Big)\Big]^{\gamma}\Big\}.
\end{aligned}$$

Next, we derive the turnpike revenue rate as the following

$$\begin{aligned}
\bar{R}(\infty) &= \lambda\bar{w}_0(\bar{f}\bar{\pi}_0^s + \bar{p}\bar{\pi}_0^p) = \lambda\bar{w}_0[\bar{f}\bar{\pi}_0^s + (\bar{f}\bar{\pi}_0^p + \bar{k}\Delta\bar{\eta}_1^w\bar{\pi}_0^p)]\\
&= \lambda\bar{w}_0\bar{f}(\bar{\pi}_0^s + \bar{\pi}_0^p) + \lambda(1 - \bar{w}_0)\Delta\bar{\eta}_1^w\bar{\pi}_1^p\\
&= \lambda\bar{w}_0[\bar{f} - \mu - (1 - \bar{\pi}_0^s - \bar{\pi}_0^p)\eta^{z\star}] + \lambda(1 - \bar{w}_0)[\bar{f} - \mu - \eta^{z\star}(1 - \pi_1^p)]\\
&= \lambda(\bar{f} - \mu) - \lambda\Big[\bar{w}_0(1 - \bar{\pi}_0^p - \bar{\pi}_0^s) + (1 - \bar{w}_0)(1 - \pi_1^p)\Big]\eta^{z\star}.
\end{aligned}$$

H. **Proof of Proposition 3.** When $\beta = 1$, we have $\eta^u_{0t} = 0$ for $t \in [0, \tau]$. Hence, we can drop U_{0t} from the steady-state equation.

When $\rho = 0$, (29) becomes $\ln(1 - \bar{\pi}^p_1) - \ln(1 - \bar{\pi}^p_0 - \bar{\pi}^s_0) = 0$, which implies that $\bar{\pi}^p_1 = \bar{\pi}^p_0 + \bar{\pi}^s_0$. Similarly, (30) implies that $\bar{\pi}^p_1 = \bar{\pi}^p_2 = \cdots = \bar{\pi}^p_{\bar{k}}$. Combining these equations with (3) gives $\Delta\bar{U}_1 = \Delta\bar{U}_2 = \cdots = \Delta\bar{U}_{\bar{k}}$, with (27) gives $\bar{w}_1 = \bar{w}_2 = \cdots = \bar{w}_{\bar{k}}$ and $\bar{\pi}^p_0 + \bar{\pi}^s_0 = \bar{\pi}^p_1 = \bar{\pi}^p_2 = \cdots = \bar{\pi}^p_{\bar{k}}$. Furthermore, by (31) and (32), we have $\bar{f} = \Delta\bar{\eta}^w_1 = \Delta\bar{\eta}^w_2 = \cdots = \Delta\bar{\eta}^w_{\bar{k}}$. Therefore, $\bar{p} = (1 + \bar{k})\bar{f}$ following (16).

Next, we expand the expressions $\bar{\pi}^p_1 = \bar{\pi}^p_0 + \bar{\pi}^s_0$ to obtain

$$\begin{aligned}
\exp\Big(\frac{a^P - \Delta\bar{U}_1}{\mu\gamma}\Big) &= \exp\Big(\frac{a - \bar{p} + \sum_{k=1}^{\bar{k}} \Delta\bar{U}_k}{\mu\gamma}\Big) + \exp\Big(\frac{a - \bar{f}}{\mu\gamma}\Big) \\
\exp\Big(\frac{a^P - a}{\mu\gamma}\Big)\exp\Big(\frac{-\Delta\bar{U}_1}{\mu\gamma}\Big) &= \exp\Big(\frac{-f - \bar{k}\bar{f} + \sum_{k=1}^{\bar{k}} \Delta\bar{U}_k}{\mu\gamma}\Big) + \exp\Big(\frac{-\bar{f}}{\mu\gamma}\Big) \\
\alpha\exp\Big(\frac{\bar{f} - \Delta\bar{U}_1}{\mu\gamma}\Big) &= \exp\Big(-\bar{k}\frac{\bar{f} - \Delta\bar{U}_1}{\mu\gamma}\Big) + 1 \qquad (44)
\end{aligned}$$

This implies that $\exp\Big(\frac{\bar{f} - \Delta\bar{U}_1}{\mu\gamma}\Big) = \phi^\alpha_{\bar{k}}$.

Next, we use the constant $\phi^\alpha_{\bar{k}}$ to determine the ratio $\bar{\pi}^p_0/\bar{\pi}^s_0$ and the population distribution $\bar{w}_0, \bar{w}_k, k = 1, \ldots, \bar{k}$. According to Equation 44, we have

$$\frac{\bar{\pi}^p_0}{\bar{\pi}^s_0} = \exp\Big(-\bar{k}\frac{\bar{f} - \Delta\bar{U}_1}{\mu\gamma}\Big) = \alpha\phi^\alpha_{\bar{k}} - 1.$$

By $\bar{w}_1 = \bar{w}_2 = \cdots = \bar{w}_{\bar{k}}$, we have $1 - \bar{w}_0 = \bar{k}\bar{w}_1$, hence

$$\frac{\bar{w}_0}{1 - \bar{w}_0} = \frac{\bar{w}_0}{\bar{k}\bar{w}_1} = \frac{\bar{\pi}^p_1}{\bar{k}\bar{\pi}^p_0} = \frac{1}{\bar{k}}\frac{\alpha\phi^\alpha_{\bar{k}}}{\alpha\phi^\alpha_{\bar{k}} - 1}$$

where the second equality follows from Equation 27, the third equality follows from Equation 44. Solving the above equation and substituting $\alpha\phi^\alpha_{\bar{k}} = (\phi^\alpha_{\bar{k}})^{-\bar{k}} + 1$ gives $\bar{w}_1 = \cdots = \bar{w}_{\bar{k}} = \Big(1 + \bar{k} + (\phi^\alpha_{\bar{k}})^{\bar{k}}\Big)^{-1}$.

Combining (34), (35) and the equations $\bar{w}_k = \bar{w}_{\bar{k}}, \bar{\pi}^p_k = \bar{\pi}^p_{\bar{k}}, \Delta\bar{\eta}^w_k = \Delta\bar{\eta}^w_{\bar{k}}, k = 1, \ldots, \bar{k} - 1$, we have $\bar{\eta}^u_1 - \bar{\eta}^u_2 = \bar{\eta}^u_2 - \bar{\eta}^u_3 = \cdots = \bar{\eta}^u_{\bar{k}-1} - \bar{\eta}^u_{\bar{k}} = \bar{\eta}^u_{\bar{k}}$, which implies that $\bar{\eta}^u_1 = \bar{k}\bar{\eta}^u_{\bar{k}}$. Then, substitute $\bar{\eta}^u_{\bar{k}}$ in (35) by $\bar{\eta}^u_1/\bar{k}$ and combine it with (36) results in the following equation

$$\begin{aligned}
\bar{w}_0\Big(\bar{f} - \eta^{z\star} - \frac{\mu}{1 - \bar{\pi}^p_0 - \bar{\pi}^s_0}\Big) &= \frac{\mu}{1 - \bar{\pi}^p_0 - \bar{\pi}^s_0}\Big[\bar{k}\frac{\bar{w}_0\bar{\pi}^p_0}{\bar{\pi}^p_1} - \frac{\bar{k}}{\mu}\bar{w}_1(1 - \bar{\pi}^p_1)(\bar{f} - \eta^{z\star})\Big] \\
\bar{w}_0(\bar{f} - \eta^{z\star}) - \bar{w}_0\Big(\frac{\mu}{1 - \bar{\pi}^p_0 - \bar{\pi}^s_0}\Big) &= \frac{\mu}{1 - \bar{\pi}^p_0 - \bar{\pi}^s_0}\bar{k}\bar{w}_1 - \bar{k}\bar{w}_1(\bar{f} - \eta^{z\star})\frac{1 - \bar{\pi}^p_1}{1 - \bar{\pi}^p_0 - \bar{\pi}^s_0} \\
\Big(\frac{\mu}{1 - \bar{\pi}^p_0 - \bar{\pi}^s_0}\Big)(\bar{w}_0 + \bar{k}\bar{w}_1) &= (\bar{f} - \eta^{z\star})(\bar{w}_0 + \bar{k}\bar{w}_1) \\
\frac{\mu}{1 - \bar{\pi}^p_0 - \bar{\pi}^s_0} &= \bar{f} - \eta^{z\star} \qquad (45)
\end{aligned}$$

substituting (45) back to (36) gives $\bar{\eta}^u_1 = 0$, which implies $\bar{\eta}^u_1 = \cdots = \bar{\eta}^u_{\bar{k}} = 0$.

Expanding (45) gives

$$\bar{f} - \eta^{z\star} = \mu\Big(1 + \Big[\exp\Big(\frac{a - (1+\bar{k})\bar{f} + \bar{k}\Delta U_1}{\mu\gamma}\Big) + \exp\Big(\frac{a-\bar{f}}{\mu\gamma}\Big)\Big]^{\gamma}\Big)$$

$$\bar{f} - \eta^{z\star} = \mu\Big(1 + \exp\Big(\frac{a-\bar{f}}{\mu}\Big)\Big[\exp\Big(-\bar{k}\frac{\bar{f}-\Delta U_1}{\mu\gamma}\Big) + 1\Big]^{\gamma}\Big)$$

$$\bar{f} - \eta^{z\star} = \mu\Big(1 + \exp\Big(\frac{a-\bar{f}}{\mu}\Big)(\phi_{\bar{k}}^{\alpha})^{\gamma}\Big),$$

which is the equation from which one can solve for the turnpike regular price.

By combining Proposition 6 and Equation (45), the turnpike revenue rate is

$$\bar{R}(0) = \lambda\bar{f}(\bar{\pi}_0^p + \bar{\pi}_0^s) = \lambda(\bar{f} - \mu) - \lambda(1 - \bar{\pi}_0^p - \bar{\pi}_0^s)\eta^{z\star}.$$

I. **Proof of Proposition 5.** This proposition is derived by combining (16), (31) and (32).

J. **Proof of Corollary 2.** This corollary follows directly from Proposition 4 and Proposition 5.

K. **The benchmark model.** We first derive the utility equation. The value of waiting is

$$u_t^0 = U_t + \epsilon_t^0,$$

the value of purchase is

$$u_t = a - f_t + \beta U_t + \epsilon_t.$$

The probability of purchase is

$$\pi_t = \Pr(u_t^0 < u_t) = \Pr(U_t + \epsilon_t^0 < a - f_t + \beta U_t + \epsilon_t).$$

If $\epsilon_t^0 - \epsilon_t$ follows logic distribution with $\mathbb{E}(\epsilon_t^0 - \epsilon_t) = 0$, the resulting choice probability is given by

$$\pi_t = \Pr(u_t^0 < u_t) = \frac{\exp(\frac{a - f_t + \beta U_t - U_t}{\mu})}{1 + \exp(\frac{a - f_t + \beta U_t - U_t}{\mu})} = \frac{\exp(\frac{a - f_t - (1-\beta)U_t}{\mu})}{1 + \exp(\frac{a - f_t - (1-\beta)U_t}{\mu})}.$$

Using arguments similar to those in Appendix A.2, we derive the utility equation as the following

$$\begin{aligned}\dot{U}_t = \lim_{\delta\to 0}\frac{U_{t+\delta} - U_t}{\delta} &= \lim_{\delta\to 0}\frac{1}{\delta}\Big(U_{t+\delta} - (1 - e^{-\lambda\delta})\big[-\ln(1-\pi_t) + U_t - e^{-\rho\delta}U_{t+\delta}\big] - e^{-\rho\delta}U_{t+\delta}\Big)\\ &= \lambda\mu\ln(1-\pi_t) + \rho U_t.\end{aligned}$$

It is more convenient to work with the new variable $U_t' \triangleq (1-\beta)U_t$ and rewrite the above equations as:

$$\pi_t = \frac{\exp\big(\frac{a - f_t - U_t'}{\mu}\big)}{1 + \exp\big(\frac{a - f_t - U_t'}{\mu}\big)}$$

and

$$\dot{U}_t' = \lambda\mu(1-\beta)\ln(1-\pi_t) + \rho U_t', \qquad U_\tau' = 0.$$

Note that if $\beta = 1$, we have $U_t' = 0$, in which case the customers behave myopically, even when $\rho = 0$.

The objective of the issuer is maximizing the total revenue over a finite horizon $[0, \tau]$, i.e.,

$$\max \int_0^\tau \lambda f_t \pi_t dt.$$

We use π_t as the control variable. The price can be expressed as a function of π_t and utility U_t', i.e.,

$$f_t = a - \mu\ln\frac{\pi_t}{1-\pi_t} - U_t'.$$

The state equations and constraints are given by

$$\begin{aligned} \dot{z}_t &= -\lambda\pi_t, & z_0 = \bar{z}, \qquad z_\tau \geqslant 0, \\ \dot{U}'_t &= \lambda\mu(1-\beta)\ln(1-\pi_t) + \rho U'_t, & U'_\tau = 0. \end{aligned}$$

The Hamiltonian is

$$H = \lambda\big(a - \mu\ln\frac{\pi_t}{1-\pi_t} - U'_t - \eta^z_t\big)\pi_t + \lambda\mu(1-\beta)\ln(1-\pi_t)\eta^u_t + \rho U'_t\eta^u_t.$$

The adjoint equations are

$$\begin{aligned} \dot{\eta}^z_t &= -\frac{\partial H}{\partial z_t} = 0 \\ \dot{\eta}^u_t &= -\frac{\partial H}{\partial U'_t} = \lambda\pi_t - \rho\eta^u_t, \ \eta^u_0 = 0 \end{aligned}$$

It is clear that the shadow price of capacity is a constant, so we can write η^z_t as η^z. Since $\lambda\pi_t \geqslant 0$, it can be easily verified that $\eta^u_t \geqslant 0$ and $\frac{\partial^2 H}{\partial \pi_t^2} \leqslant 0$. So the Hamiltonian is concave in π_t.

Lemma 7. *The steady-state equations for the optimal policy are:*

$$\begin{aligned} \rho\bar{\eta}^u &= \lambda\bar{\pi} \\ \rho\bar{U}' &= -\lambda\mu(1-\beta)\ln(1-\bar{\pi}) \\ \mu(1-\beta)\frac{\bar{\eta}^u}{1-\bar{\pi}} &= -\mu\big(1+\frac{\bar{\pi}}{1-\bar{\pi}}\big) + a - \mu\ln\frac{\bar{\pi}}{1-\bar{\pi}} - \bar{U}' - \eta^z. \end{aligned}$$

where the third equation is derived from the first order optimality condition.

L. **Proof of Lemma 5.** In Section K, we have shown that if $\beta = 1$, the fully strategic customers (with $\rho = 0$) have the same behaviors as myopic customers (with $\rho = \infty$). We first derive the optimal price for these customers in the benchmark model.

For the case of $\eta^{z\star} = 0$, the optimal price f_n satisfy the following equation

$$f_n = \mu\Big[1 + \exp\Big(\frac{a - f_n}{\mu}\Big)\Big], \tag{46}$$

with the revenue rate $\bar{R}^n(0) = \bar{R}^n(\infty) = \lambda(f_n - \mu)$.

Next, we examine the case of selling passes to myopic customers. According to Proposition 2, the optimal price $\bar{f}_m$ satisfies the equation

$$\bar{f}_m = \mu\Big\{1 + \Big[\exp\Big(\frac{a - \bar{f}_m}{\mu\gamma}\Big) + \exp\Big(\frac{a - \bar{f}_m - \bar{k}(\bar{f}_m - \mu)(1 + e^{-a^P/\mu}) + \bar{k}\eta^{z\star}e^{-a^P/\mu}}{\mu\gamma}\Big)\Big]^\gamma\Big\}. \tag{47}$$

The corresponding turnpike revenue rate is $\bar{R}(\infty) = \lambda(\bar{f}_m - \mu)$ for the case of $\eta^{z\star} = 0$.

Since the RHS of equation (47) is greater than the RHS of equation (46) but the revenue rates have the same expression, we can conclude that $f_n < \bar{f}_m$ and hence $\bar{R}^n(\infty) < \bar{R}(\infty)$.

Next, we turn to the case of selling passes to fully strategic customers. When $\beta = 1$, according to Proposition 3, the turnpike regular price $\bar{f}_s$ for the case of $\eta^{z\star} = 0$ is determined by the equation

$$\bar{f}_s = \mu\Big(1 + \exp\Big(\frac{a - \bar{f}_s}{\mu}\Big)(\phi^\alpha_{\bar{k}})^\gamma\Big), \tag{48}$$

with the corresponding turnpike revenue rate $\bar{R}(0) = \lambda(\bar{f}_s - \mu)$. Note that $\phi^\alpha_{\bar{k}} > 1$, so the RHS of (48) is greater than the RHS of (46). So $\bar{R}^n(0) < \bar{R}(0)$ following the same argument as above.

M. **Proof of Lemma 6.** For $\rho > 0$, let $\bar{x}(\rho)$ be the solution to the following equation

$$\mu(1+x) + \frac{\lambda\mu(1-\beta)}{\rho}\ln(1+x) = a - \mu\ln x - \frac{\lambda\mu(1-\beta)}{\rho}x. \tag{49}$$

It can be verified that the turnpike price and revenue rate (under ample capacity, i.e., $\eta^z = 0$) are given by

$$\bar{f} = \mu[1+\bar{x}(\rho)] + \frac{\lambda\mu(1-\beta)}{\rho}\bar{x}(\rho), \qquad \bar{R}(\rho) = \lambda\mu(1-\beta)\bar{x}(\rho)\Big(1 + \frac{\lambda}{\rho}\frac{\bar{x}(\rho)}{1+\bar{x}(\rho)}\Big).$$

Under ample capacity, we have $\eta^z = 0$. Then the equation (49) follows from combining the three equations in Lemma 7 and letting $x = \bar{\pi}/(1-\bar{\pi})$. For convenience, we write $\bar{x}(\rho)$ as $\bar{x}$. Note that

$$\begin{aligned}
\bar{f} &= a - \mu\ln(\frac{\bar{\pi}}{1-\bar{\pi}}) - \bar{U} \\
&= a - \mu\ln(\frac{\bar{\pi}}{1-\bar{\pi}}) + \frac{\lambda\mu(1-\beta)}{\rho}\ln(1-\bar{\pi}) \\
&= a - \mu\ln\bar{x} - \frac{\lambda\mu(1-\beta)}{\rho}\ln(1+\bar{x}) \\
&= \mu(1+\bar{x}) + \frac{\lambda\mu(1-\beta)}{\rho}\bar{x},
\end{aligned}$$

where the last equation follows from (49). Therefore, the turnpike revenue rate can be expressed as a function of $\bar{x}$ and ρ in the following

$$\bar{R}(\rho) = \lambda\bar{f}\bar{\pi} = \lambda\mu(1-\beta)\Big(\mu(1+\bar{x}) + \frac{\lambda\mu(1-\beta)}{\rho}\bar{x}\Big)\frac{\bar{x}}{1+\bar{x}} = \lambda\mu(1-\beta)\bar{x}\Big(1 + \frac{\lambda}{\rho}\frac{\bar{x}}{1+\bar{x}}\Big). \tag{50}$$

Next, we show that $\bar{R}(\rho)$ is increasing in ρ in two steps: the first step shows that $\bar{x}$ is increasing in ρ, the second step shows that $\bar{R}$ is increasing in $\bar{x}$. By equation (49), we obtain

$$\frac{\lambda}{\rho} = \frac{a - \mu\ln\bar{x} - \mu(1+\bar{x})}{\mu(1-\beta)[\bar{x} + \ln(1+\bar{x})]}, \tag{51}$$

from which we obtain $\partial\bar{x}/\partial\rho > 0$.

Now we move to the second step. By substituting (51) into (50), the revenue rate can be expressed as a function of $\bar{x}$, i.e.,

$$\bar{R}(\bar{x}) = \lambda\mu(1-\beta)\bar{x}\Big(1 + \frac{\bar{x}}{1+\bar{x}}\frac{a - \mu\ln\bar{x} - \mu(1+\bar{x})}{\mu[\bar{x} + \ln(1+\bar{x})]}\Big)$$

The non-negativity of $\bar{f}$ and $\bar{U}$ implies that

$$\begin{aligned}
&a - \mu\ln(\frac{\bar{\pi}}{1-\bar{\pi}}) \geqslant 0 \\
&a - \mu\ln\bar{\pi} \geqslant -\mu\ln(1-\bar{\pi}) \geqslant \mu\bar{\pi},
\end{aligned}$$

the last inequality follows from the log inequality.

Next, we will use the above inequality to bound the derivative of revenue rate with respect to $\bar{x}$:

$$\begin{aligned}
\frac{\partial \bar{R}(\bar{x})}{\partial \bar{x}} &= \lambda\mu(1-\beta)\frac{\ln(1+\bar{x})}{(1+\bar{x})^2[\bar{x}+\ln(1+\bar{x})]^2}\Big\{\mu(1+\bar{x})^2\ln(1+\bar{x}) + \big[-\mu(2+\bar{x})\ln\bar{x} + (a-\mu)\bar{x} + 2a - \mu\big]\bar{x}\Big\} \\
&= \lambda\mu(1-\beta)\frac{\ln(1+\bar{x})}{(1+\bar{x})^2[\bar{x}+\ln(1+\bar{x})]^2}\Big\{\mu(1+\bar{x})^2\ln(1+\bar{x}) + \big[(2+\bar{x})(a-\mu\ln x) - (1+\bar{x})\mu\big]\bar{x}\Big\} \\
&\geqslant \lambda\mu(1-\beta)\frac{\ln(1+\bar{x})}{(1+\bar{x})^2[\bar{x}+\ln(1+\bar{x})]^2}\Big\{\mu\ln(1+\bar{x}) + \big[(2+\bar{x})\mu\bar{\pi} - (1+\bar{x})\mu\big]\bar{x}\Big\} \\
&= \lambda\mu(1-\beta)\frac{\ln(1+\bar{x})}{(1+\bar{x})^2[\bar{x}+\ln(1+\bar{x})]^2}\Big\{\mu\ln(1+\bar{x}) - \mu\bar{\pi}\Big\} \\
&\geqslant \lambda\mu(1-\beta)\frac{\ln(1+\bar{x})}{(1+\bar{x})^2[\bar{x}+\ln(1+\bar{x})]^2}\Big\{\mu\frac{\bar{x}}{1+\bar{x}} - \mu\bar{\pi}\Big\} \\
&= 0.
\end{aligned}$$

Therefore,

$$\frac{\partial \bar{R}(\bar{x})}{\partial \rho} = \frac{\partial \bar{R}(\bar{x})}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial \rho} \geqslant 0.$$